The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM)
ISSN: 2602-3199

The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM), 2022

Volume 18, Pages 7-15

ICBASET 2022: International Conference on Basic Sciences, Engineering and Technology

On a Geometric Programming Approach to Profit Maximization: The Case of CES Technology

Vedran KOJIĆ
University of Zagreb

Abstract: The profit maximization problem takes a central place in the theory of the firm, especially when conditions for perfect competition hold. In this paper, we solve the profit maximization problem of a perfectly competitive firm when the constant elasticity of substitution (CES) production function with $n \geq 2$ inputs describes its technology. Commonly, this problem is solved by using multivariable differential calculus. However, to avoid tedious algebraic manipulations and bypass checking nontrivial necessary and sufficient conditions, we employ geometric programming (GP), and the power mean inequality (PMI) as an elegant complementary tool to multivariable calculus. Since the GP and the PMI are simple optimization techniques without derivatives, they can provide new insights into the given problem to managers, students, and other audiences who may be unfamiliar with multivariable differential calculus. Additionally, by using the properties of limits, we show that the solution to the profit maximization problem with Cobb-Douglas technology is a limiting case of our result.

Keywords: Profit maximization, Cobb-Douglas technology, CES technology, Geometric programming

Introduction

Since finding the optimum using differential calculus may be nontrivial and inelegant, the signomial geometric programming (SGP) technique with zero degrees of difficulty was proposed by (Liu, 2006) as a complementary approach to solving the profit maximization problem with Cobb-Douglas production function (CDPF). Further, (Kojić & Lukač, 2018) showed how the results from (Liu, 2006) can be obtained by solving an equivalent geometric programming (GP) problem with zero degrees of difficulty, providing proof that the (global) maximum profit in the case of CDPF has been achieved.

In this note, we solve a profit maximization problem where the firm's technology is given by the CES production function (CESPF). Furthermore, since the CESPF is a generalization of the CDPF, we show how the results by (Liu, 2006) and (Kojić & Lukač, 2018) can be derived from our results.

The paper is organized as follows. After the introduction, the notation and preliminaries are presented in the second section. The solution to the profit maximization problem with CESPF is the main result of the paper and it is given in the third section. The fourth section shows that the solution to the profit maximization problem with CDPF is a limiting case of our result. The fifth section concludes the paper.

Notations and preliminaries

Notations have been adopted from (Avvakumov et al., 2010), (Jehle & Reny, 2011), and (Liu, 2006):

p	market price per unit, $p>0$
A	scale of production, $A>0$
x_i	input quantities, $x_i>0$
v_i	input prices, $v_i>0$ for all i
φ_i	elasticities of Cobb-Douglas production function (CDPF) $\varphi_i>0$ for all i
x_i^{C-D}	the i^{th} component of the maximizer
$\pi^{C\text{-}D}$	maximum profit in the case of CDPF (i.e. Cobb-Douglas technology)
α_i	allocation coefficients of CES production function (CESPF, i.e. CES technology), $\alpha_i>0$ for all i
σ	degree of homogeneity of CESPF
ρ	substitution coefficient of CESPF
x_i^{CES}	the i^{th} component of the maximizer
π^{CES}	maximum profit in the case of CESPF

Considering CDPF with n inputs ($n\geq 2$), given by

$$f\left(x_1, x_2, \ldots, x_n\right) = A\prod_{i=1}^{n} x_i^{\varphi_i} \;, \tag{1}$$

the profit maximization problem becomes

$$\pi^{C-D} = \max_{x_1, x_2, \ldots, x_n > 0} p\left(A\prod_{i=1}^{n} x_i^{\varphi_i} \right) - \sum_{i=1}^{n} v_i x_i \;, \tag{2}$$

where

$$\sum_{i=1}^{n} \varphi_i < 1 \,. \tag{3}$$

By using SGP, (Liu, 2006) obtained the result of problem (2) as follows:

$$\pi^{C-D} = \left(1 - \sum_{i=1}^{n} \varphi_i\right) \left(pA\right)^{1\Big/\left(1-\sum_{i=1}^{n}\varphi_i\right)} \prod_{i=1}^{n} \left(\frac{v_i}{\varphi_i}\right)^{-\varphi_i\Big/\left(1-\sum_{i=1}^{n}\varphi_i\right)} , \tag{4}$$

$$x_i^{C-D} = \frac{\pi^{C-D}\varphi_i}{v_i\left(1 - \sum_{j=1}^{n}\varphi_j\right)} \,, \tag{5}$$

for all i=1, 2, …, n. However, problem (2) can be converted into a GP problem with zero degrees of difficulty, and the same result (4)-(5) was obtained by (Kojić & Lukač, 2018).
Now, let us introduce CESPF as in (Avvakumov et al., 2010), defined by

$$\psi\left(x_1, x_2, \ldots, x_n\right) = A\left(\sum_{i=1}^{n} \alpha_i x_i^{-\rho}\right)^{-\sigma/\rho} \;, \tag{6}$$

Where

$$\sum_{i=1}^{n} \alpha_i = 1 \,, \tag{7}$$

$$\sigma > 0 \,, \tag{8}$$

$$\rho > -1, \; \rho \neq 0 \,. \tag{9}$$

Considering CESPF, the profit maximization problem of a perfectly competitive firm with CES technology becomes (Jehle & Reny, 2011):

$$\begin{aligned} \pi^{CES} = \max_{y, x_1, x_2, \ldots, x_n > 0} & \; py - \sum_{i=1}^{n} v_i x_i \\ \text{s.t. } & A\left(\sum_{i=1}^{n} \alpha_i x_i^{-\rho} \right)^{-\sigma/\rho} \geq y. \end{aligned} \tag{10}$$

To solve (10), we will use a power mean inequality (see (Bullen, 2003)).

Lemma 1. (Power mean inequality) Let $n \geq 2$, $n \in \mathbb{N}$, $x_i > 0$, and $w_i > 0$, $i=1, 2, \ldots, n$, such that $w = \sum_{i=1}^{n} w_i$. Then, for all $r>0$, the following inequality holds

$$\prod_{i=1}^{n} x_i^{w_i} \leq \left(\sum_{i=1}^{n} \frac{w_i}{w} x_i^r \right)^{w/r} . \tag{11}$$

Equality in (11) holds if and only if $x_1 = x_2 = \ldots = x_n$.

The profit maximization problem with CES technology

Since the very discovery of geometric programming, it has been applied to various optimization problems in science and industry. In the economic theory of production, (Reklaitis et al., 1975) applied geometric programming to solve the firm's cost minimization problem with Cobb-Douglas and CES production functions. Mainly, they used power mean inequality in solving the cost minimization problem with CESPF using geometric programming. However, they didn't comment on the sign of the substitution coefficient ρ, which makes their analysis incomplete.

Another important problem in the theory of the firm is the profit maximization problem for the perfectly competitive firm. The profit maximization problems with Cobb-Douglas and CES production function and their solutions are very well known in the literature (Zevelev, 2014). These problems are usually solved using multivariable calculus (Avvakumov et al., 2010). Still, a geometric programming approach to the profit maximization problem with CESPF is the main result of this paper since, to the best of our knowledge, such an approach is unknown in the literature.

According to (Duffin et al., 1967), (Beightler & Philips, 1976) and (Boyd et al., 2007), problem (10) is equivalent to the following problem:

$$\max_{z, y, x_1, \ldots, x_n > 0} \quad z \tag{12}$$

$$\text{s.t.} \quad py - \sum_{i=1}^{n} v_i x_i \geq z , \tag{13}$$

$$A\left(\sum_{i=1}^{n} \alpha_i x_i^{-\rho} \right)^{-\sigma/\rho} \geq y \ . \tag{14}$$

Since max f = 1/(min (1/f)) for a positive function f, instead of the problem (12)-(14) we will solve the problem (15)-(17):

$$\min_{z, y, x_1, \ldots, x_n > 0} \quad z^{-1} \tag{15}$$

$$\text{s.t.} \quad p^{-1} y^{-1} z + \sum_{i=1}^{n} p^{-1} v_i x_i y^{-1} \leq 1 , \tag{16}$$

$$A\left(\sum_{i=1}^{n} \alpha_i x_i^{-\rho} \right)^{-\sigma/\rho} \geq y \ . \tag{17}$$

Let us consider two cases regarding the value of the substitution coefficient of CESPF ρ from (9).

Case 1. ρ>0

Since the degree of homogeneity σ is defined as a positive number (see (8)), and when ρ>0, the function $x \mapsto x^{-\rho/\sigma}$ is decreasing (for all x>0). In this case, the inequality (17) is equivalent to

$$A^{-\frac{\rho}{\sigma}}\left(\sum_{i=1}^{n}\alpha_i x_i^{-\rho}\right) \le y^{-\frac{\rho}{\sigma}} \;\Leftrightarrow\; \sum_{i=1}^{n} A^{-\frac{\rho}{\sigma}}\alpha_i x_i^{-\rho} y^{\frac{\rho}{\sigma}} \le 1\,, \tag{18}$$

which transforms the problem (15)-(17) into a GP problem (15)-(16) and (18) written in standard form. According to (Boyd et al., 2007) and (Duffin et al., 1967), the corresponding dual of (15)-(16) and (18) is the following problem:

$$M = \max_{\substack{\beta,\gamma,\delta_i,\varepsilon_i>0\\ i=1,2,\dots,n}} \left(\frac{1}{\beta}\right)^{\beta}\cdot\left(\frac{p^{-1}}{\gamma}\right)^{\gamma}\cdot\prod_{i=1}^{n}\left(\left(\frac{p^{-1}v_i}{\delta_i}\right)^{\delta_i}\left(\frac{A^{-\rho/\sigma}\alpha_i}{\varepsilon_i}\right)^{\varepsilon_i}\right)\cdot\left(\gamma+\sum_{i=1}^{n}\delta_i\right)^{\gamma+\sum_{i=1}^{n}\delta_i}\cdot\left(\sum_{i=1}^{n}\varepsilon_i\right)^{\sum_{i=1}^{n}\varepsilon_i}, \tag{19}$$

s.t. β=1

$$\begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & -1 & \cdots & -1 & \rho/\sigma & \rho/\sigma & \cdots & \rho/\sigma \\ 0 & 0 & 1 & 0 & \cdots & 0 & -\rho & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & -\rho & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & -\rho \end{bmatrix}_{(n+2)\times(2n+2)} \cdot \begin{bmatrix} \beta \\ \gamma \\ \delta_1 \\ \vdots \\ \delta_n \\ \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}_{(2n+2)\times 1} = \mathbf{0}, \tag{20}$$

where a null vector **0** in (20) has n+2 components, and β>0, γ>0, δ_i>0, ε_i>0, i = 1, 2, …, n, are dual variables. From (20) we get

$$\beta = \gamma = 1\,, \tag{21}$$

$$\varepsilon_i = \frac{\delta_i}{\rho},\; i = 1,2,\dots,n\,, \tag{22}$$

$$\sum_{i=1}^{n}\delta_i = \frac{\sigma}{1-\sigma}\,, \tag{23}$$

$$\sum_{i=1}^{n}\varepsilon_i = \frac{\sigma}{\rho(1-\sigma)}\,. \tag{24}$$

Note that from (8), (23) and the positivity of dual variables, it follows that the degree of homogeneity of CESPF must satisfy the following condition:

$$0 < \sigma < 1\,. \tag{25}$$

Furthermore, using (21)-(24), (19)-(20) becomes an unconstrained maximization problem

$$M = \max_{\substack{\delta_i>0\\ i=1,2,\dots,n}} \left(\frac{p^{-1}A^{-1}}{1-\sigma}\right)^{\frac{1}{1-\sigma}}\left(\frac{\sigma}{1-\sigma}\right)^{\frac{\sigma}{\rho(1-\sigma)}}\prod_{i=1}^{n}\left(\frac{v_i\alpha_i^{1/\rho}}{\delta_i^{(\rho+1)/\rho}}\right)^{\delta_i}\,. \tag{26}$$

Let us solve (26) by using Lemma 1. Let

$$x_i = \frac{v_i \alpha_i^{1/\rho}}{\delta_i^{(\rho+1)/\rho}}, \; w_i = \delta_i, \; i = 1,2,...,n, \tag{27}$$

$$w = w_1 + w_2 + \cdots + w_n = \sum_{i=1}^{n} \delta_i \overset{(23)}{=} \frac{\sigma}{1-\sigma}. \tag{28}$$

According to (11), for all r>0, the following inequality holds

$$\prod_{i=1}^{n} \left(\frac{v_i \alpha_i^{1/\rho}}{\delta_i^{(\rho+1)/\rho}} \right)^{\delta_i} \overset{(11)}{\le} \left(\sum_{i=1}^{n} \frac{\delta_i}{\delta_1 + \cdots \delta_n} \left(\frac{v_i \alpha_i^{1/\rho}}{\delta_i^{(\rho+1)/\rho}} \right)^{r} \right)^{\frac{\delta_1 + \cdots \delta_n}{r}} = \left(\frac{1-\sigma}{\sigma} \right)^{\frac{\sigma}{r(1-\sigma)}} \left(\sum_{i=1}^{n} \left(v_i \alpha_i^{1/\rho} \right)^{r} \delta_i^{1 - r \cdot \frac{\rho+1}{\rho}} \right)^{\frac{\sigma}{r(1-\sigma)}}. \tag{29}$$

By choosing $r=\rho/(\rho+1)$, the right-hand side of (29) becomes the constant. Thus, from (26) and (29), we get

$$\left(\frac{p^{-1} A^{-1}}{1-\sigma} \right)^{\frac{1}{1-\sigma}} \left(\frac{\sigma}{1-\sigma} \right)^{\frac{\sigma}{\rho(1-\sigma)}} \prod_{i=1}^{n} \left(\frac{v_i \alpha_i^{1/\rho}}{\delta_i^{(\rho+1)/\rho}} \right)^{\delta_i} \le \left(\frac{p^{-1} A^{-1}}{1-\sigma} \right)^{\frac{1}{1-\sigma}} \left(\frac{\sigma}{1-\sigma} \right)^{\frac{-\sigma}{1-\sigma}} \left(\sum_{i=1}^{n} v_i^{\frac{\rho}{\rho+1}} \alpha_i^{\frac{1}{\rho+1}} \right)^{\frac{(\rho+1)\sigma}{\rho(1-\sigma)}}. \tag{30}$$

Equality in (29)-(30) holds if and only if

$$\frac{v_i \alpha_i^{1/\rho}}{\delta_i^{(\rho+1)/\rho}} = \frac{v_1 \alpha_1^{1/\rho}}{\delta_1^{(\rho+1)/\rho}}, \; i = 1,2,...,n. \tag{31}$$

From (23) and (31) we get

$$\delta_i = \frac{\frac{\sigma}{1-\sigma} v_i^{\frac{\rho}{\rho+1}} \alpha_i^{\frac{1}{\rho+1}}}{\sum_{j=1}^{n} v_j^{\frac{\rho}{\rho+1}} \alpha_j^{\frac{1}{\rho+1}}}. \tag{32}$$

Thus, by definition of the strict global optimum, the strict global maximum M of (26), and at the same time of (19), is equal to the right-hand side of (30), and it is achieved if and only if δ_i, i=1,2,…,n, satisfy (32). In addition, M represents the strict global minimum of (15)-(17). Furthermore, since max f = 1/(min (1/f)) for a positive function f, we can find π^{CES} from (10) via (12)-(17) as follows:

$$\pi^{CES} = M^{-1} = \left(pA(1-\sigma) \right)^{1/(1-\sigma)} \left(\frac{\sigma}{1-\sigma} \right)^{\frac{\sigma}{1-\sigma}} \left(\sum_{i=1}^{n} v_i^{\frac{\rho}{1+\rho}} \alpha_i^{\frac{1}{1+\rho}} \right)^{\frac{-(1+\rho)\sigma}{\rho(1-\sigma)}}. \tag{33}$$

According to (Duffin et al., 1967), from (15)-(18) and (21)-(24), we have

$$z^{-1} = \beta M = M \;\; \Rightarrow \;\; z = M^{-1}, \tag{34}$$

$$p^{-1} y^{-1} z = \frac{\gamma}{\gamma + \sum_{j=1}^{n} \delta_j} = \frac{1}{1 + \sum_{j=1}^{n} \delta_j}, \tag{35}$$

$$p^{-1} v_i x_i y^{-1} = \frac{\delta_i}{\gamma + \sum_{j=1}^{n} \delta_j}, \; i = 1,2,...,n, \tag{36}$$

$$A^{-\rho/\sigma} \alpha_i x_i^{-\rho} y^{\rho/\sigma} = \frac{\varepsilon_i}{\sum_{j=1}^{n} \varepsilon_j}, \; i = 1,2,...,n, \tag{37}$$

from where we get

$$x_i^{CES} = \frac{\frac{\sigma}{1-\sigma}\left(\frac{\alpha_i}{v_i}\right)^{\frac{1}{\rho+1}}}{\sum_{k=1}^{n} v_k^{\frac{\rho}{\rho+1}} \alpha_k^{\frac{1}{\rho+1}}} \cdot \pi^{CES},\ i = 1, 2, ..., n\ . \tag{38}$$

Thus, the strict global maximum of the profit maximization problem with CES technology π^{CES} is given by (33), and it is achieved for the input values given by (38).

Case 2. -1<ρ<0

If -1<ρ<0, then the function $x \mapsto x^{-\rho/\sigma}$ is increasing (for all x>0). In this case, inequality (17) is equivalent to

$$A^{-\frac{\rho}{\sigma}}\left(\sum_{i=1}^{n} \alpha_i x_i^{-\rho}\right) \geq y^{-\frac{\rho}{\sigma}} \iff -\sum_{i=1}^{n} A^{-\frac{\rho}{\sigma}} \alpha_i x_i^{-\rho} y^{\frac{\rho}{\sigma}} \leq -1, \tag{39}$$

so (15)-(17) becomes a signomial geometric programming problem (15)-(16) and (39). SGP problem (15)-(16) and (39) can be solved similarly to (15)-(16) and (18). The only difference is changing parameter ρ<0 by $-\rho$>0 into equations (19)-(24). From that, by doing some algebraic calculations, it follows that the solution in this case is

$$\pi^{CES} = \left(pA(1-\sigma)\right)^{1/(1-\sigma)} \left(\frac{\sigma}{1-\sigma}\right)^{\frac{\sigma}{1-\sigma}} \left(\sum_{i=1}^{n} v_i^{\frac{-\rho}{1-\rho}} \alpha_i^{\frac{1}{1-\rho}}\right)^{\frac{-(1-\rho)\sigma}{-\rho(1-\sigma)}}, \tag{40}$$

$$x_i^{CES} = \frac{\frac{\sigma}{1-\sigma}\left(\frac{\alpha_i}{v_i}\right)^{\frac{1}{-\rho+1}}}{\sum_{k=1}^{n} v_k^{\frac{-\rho}{-\rho+1}} \alpha_k^{\frac{1}{-\rho+1}}} \cdot \pi^{CES},\ i = 1, 2, ..., n\ . \tag{41}$$

Note that (40) and (41) have the same form as (33) and (38), respectively, where the only difference is the change of parameter ρ with $-\rho$.

The profit maximization problem with Cobb-Douglas technology

In this section, we show that the solution to the profit maximization problem with Cobb-Douglas technology given by (4)-(5) is the limiting case of (33) and (38) when ρ>0, i.e. the limiting case of (40)-(41) when -1<ρ<0. Let us first show how the Cobb-Douglas production function can be obtained from the CES production function. Let CDPF be given by (1) where (3) holds. Let us define

$$\sigma = \sum_{i=1}^{n} \varphi_i\ , \tag{42}$$

$$\alpha_i = \frac{\varphi_i}{\sigma},\ i = 1, 2, ..., n\ . \tag{43}$$

Note that (3) and (42) imply (25). Furthermore, note that

$$\sum_{i=1}^{n} \alpha_i \overset{(43)}{=} \sum_{i=1}^{n} \frac{\varphi_i}{\sigma} = \frac{1}{\sigma}\sum_{i=1}^{n} \varphi_i \overset{(42)}{=} 1\ . \tag{44}$$

Let

$$U = \lim_{\rho \to 0} A \left(\sum_{j=1}^{n} \alpha_j x_j^{-\rho} \right)^{-\frac{\sigma}{\rho}}. \tag{45}$$

Since (44) holds, by taking a natural logarithm and after applying L'Hospital's rule, from (45) we have

$$\begin{aligned} \ln U &= \ln A + \lim_{\rho \to 0} \frac{-\sigma \ln \left(\sum_{i=1}^{n} \alpha_i x_i^{-\rho} \right)}{\rho} \overset{L'H}{=} \ln A + \lim_{\rho \to 0} \frac{\sigma \sum_{i=1}^{n} \left(x_i^{-\rho} \ln x_i^{\alpha_i} \right)}{\sum_{i=1}^{n} \alpha_i x_i^{-\rho}} = \\ &= \ln A + \frac{\sigma \sum_{i=1}^{n} \ln x_i^{\alpha_i}}{\sum_{i=1}^{n} \alpha_i} = \ln A + \frac{\sum_{i=1}^{n} \ln x_i^{\sigma\alpha_i}}{1} = \ln \left(A \prod_{i=1}^{n} x_i^{\sigma\alpha_i} \right) \overset{(43)}{=} \ln \left(A \prod_{i=1}^{n} x_i^{\varphi_i} \right). \end{aligned} \tag{46}$$

Then, from (46) we have

$$U = \lim_{\rho \to 0} A \left(\sum_{i=1}^{n} \alpha_i x_i^{-\rho} \right)^{-\frac{\sigma}{\rho}} = A \prod_{i=1}^{n} x_i^{\varphi_i}. \tag{47}$$

Thus, the Cobb-Douglas production function given by (1) is the limit when $\rho \to 0$ of the CES production function given by (6).

Further, let us show how $\pi^{C\text{-}D}$ from (4) can be obtained from π^{CES}. From (33) we have

$$\begin{aligned} \lim_{\rho \to 0^+} \pi^{CES} &= \lim_{\rho \to 0} \left(pA(1-\sigma) \right)^{\frac{1}{1-\sigma}} \left(\frac{\sigma}{1-\sigma} \right)^{\frac{\sigma}{1-\sigma}} \left(\sum_{i=1}^{n} v_i^{\frac{\rho}{\rho+1}} \alpha^{\frac{1}{\rho+1}} \right)^{\frac{-(\rho+1)\sigma}{\rho(1-\sigma)}} \\ &= \left(pA(1-\sigma) \right)^{\frac{1}{1-\sigma}} \left(\frac{\sigma}{1-\sigma} \right)^{\frac{\sigma}{1-\sigma}} \cdot \Lambda, \end{aligned} \tag{48}$$

where

$$\Lambda = \lim_{\rho \to 0^+} \left(\sum_{i=1}^{n} v_i^{\frac{\rho}{\rho+1}} \alpha^{\frac{1}{\rho+1}} \right)^{\frac{-(1+\rho)\sigma}{\rho(1-\sigma)}}. \tag{49}$$

Since (44) holds, by taking a natural logarithm and after applying L'Hospital's rule, from (49) we have

$$\begin{aligned} \ln \Lambda &= \lim_{\rho \to 0^+} \frac{-(\rho+1)\sigma \ln \left(\sum_{i=1}^{n} v_i^{\frac{\rho}{\rho+1}} \alpha_i^{\frac{1}{\rho+1}} \right)}{\rho(1-\sigma)} = \\ &\overset{L'H}{=} \lim_{\rho \to 0^+} \frac{-\sigma \ln \left(\sum_{i=1}^{n} v_i^{\frac{\rho}{\rho+1}} \alpha_i^{\frac{1}{\rho+1}} \right) - (\rho+1)\sigma \left(\frac{1}{(\rho+1)^2} \sum_{i=1}^{n} v_i^{\frac{\rho}{\rho+1}} \alpha_i^{\frac{1}{\rho+1}} \ln \frac{v_i}{\alpha_i} \right) \Big/ \left(\sum_{i=1}^{n} v_i^{\frac{\rho}{\rho+1}} \alpha_i^{\frac{1}{\rho+1}} \right)}{1-\sigma} \\ &= -\frac{\sigma}{1-\sigma} \sum_{i=1}^{n} \alpha_i \ln \frac{v_i}{\alpha_i} = \ln \prod_{i=1}^{n} \left(\frac{v_i}{\alpha_i} \right)^{-\frac{\sigma\alpha_i}{1-\sigma}}. \end{aligned} \tag{50}$$

Thus, from (42)-(43) and (48)-(50) we have

$$\begin{aligned}\lim_{\rho\to 0^+}\pi^{CES} &= \left(pA(1-\sigma)\right)^{\frac{1}{1-\sigma}}\left(\frac{\sigma}{1-\sigma}\right)^{\frac{\sigma}{1-\sigma}}\prod_{i=1}^{n}\left(\frac{v_i}{\alpha_i}\right)^{-\frac{\sigma\alpha_i}{1-\sigma}}\\ &= (pA)^{1/\left(1-\sum_{i=1}^n\varphi_i\right)}(1-\sigma)\sigma^{\frac{\sigma}{1-\sigma}}\prod_{i=1}^{n}\left(\frac{v_i}{\varphi_i/\sigma}\right)^{\frac{-\varphi_i}{1-\sum_{i=1}^n\varphi_i}}\\ &= (pA)^{1/\left(1-\sum_{i=1}^n\varphi_i\right)}(1-\sigma)\sigma^{\frac{\sigma}{1-\sigma}}\sigma^{\frac{-\sigma}{1-\sigma}}\prod_{i=1}^{n}\left(\frac{v_i}{\varphi_i}\right)^{\frac{-\varphi_i}{1-\sum_{i=1}^n\varphi_i}}\\ &= (pA)^{1/\left(1-\sum_{i=1}^n\varphi_i\right)}\left(1-\sum_{i=1}^n\varphi_i\right)\prod_{i=1}^{n}\left(\frac{v_i}{\varphi_i}\right)^{\frac{-\varphi_i}{1-\sum_{i=1}^n\varphi_i}}\\ &= \pi^{C-D}.\end{aligned} \tag{51}$$

Thus, the profit $\pi^{C\text{-}D}$ from (4) is the limit when $\rho\to 0^+$ of the profit π^{CES} from (33). When $-1<\rho<0$, by using a similar procedure, we can show that the profit $\pi^{C\text{-}D}$ from (4) is the limit when $\rho\to 0^-$ of the profit π^{CES} from (40).

Finally, let us show how x_i^{C-D} from (5) can be obtained from x_i^{CES}, $i=1,2,...,n$. Let

$$V_i = \lim_{\rho\to 0^+} x_i^{CES},\ i=1,2,...,n. \tag{52}$$

Since (43)-(44) hold, from (38), (51) and (52) we have

$$\begin{aligned}V_i &= \lim_{\rho\to 0^+}\frac{\frac{\sigma}{1-\sigma}\left(\frac{\alpha_i}{v_i}\right)^{\frac{1}{\rho+1}}}{\sum_{k=1}^{n}v_k^{\frac{\rho}{\rho+1}}\alpha_k^{\frac{1}{\rho+1}}}\cdot\pi^{CES} = \lim_{\rho\to 0^+}\frac{\frac{\sigma}{1-\sigma}\left(\frac{\alpha_i}{v_i}\right)^{\frac{1}{\rho+1}}}{\sum_{k=1}^{n}v_k^{\frac{\rho}{\rho+1}}\alpha_k^{\frac{1}{\rho+1}}}\cdot\lim_{\rho\to 0^+}\pi^{CES} =\\ &= \frac{\frac{\sigma}{1-\sigma}\cdot\frac{\alpha_i}{v_i}}{\sum_{k=1}^{n}1\cdot\alpha_k^1}\cdot\pi^{C-D} = \frac{\sigma}{1-\sigma}\cdot\frac{\varphi_i/\sigma}{v_i}\cdot\pi^{C-D} = \frac{\pi^{C-D}\varphi_i}{v_i\left(1-\sum_{k=1}^n\varphi_k\right)}\\ &= x_i^{C-D}.\end{aligned} \tag{53}$$

Thus, x_i^{C-D} from (5) is the limit when $\rho\to 0^+$ of x_i^{CES}, $i=1,2,...,n$, from (38). When $-1<\rho<0$, by using a similar procedure, we can show that x_i^{C-D} from (5) is the limit when $\rho\to 0^-$ of x_i^{CES}, $i=1,2,...,n$ from (41).

Conclusion

One of the most important problems in economics is the firm's profit maximization problem. In this paper, we solved the profit maximization problem with CES technology with $n\geq 2$ inputs using a geometric programming approach and power mean inequality. Unlike multivariable calculus, this procedure does not require checking nontrivial necessary and sufficient conditions. Instead, a unique solution followed by power mean inequality and the definition of a strict global maximum directly. Finally, by using L'Hospital rule only, we showed how the solution to the profit maximization problem with Cobb-Douglas technology could be derived from the CES technology case.

Recommendations

Given that the geometric programming approach to the profit maximization problem bypasses checking nontrivial necessary and sufficient conditions when multivariable calculus is used, educators could incorporate this technique into classrooms to present this important topic in a complementary way. Since geometric programming and the power mean inequality are simple optimization techniques without derivatives, they can provide new insights into the given problem to managers, students, and other audiences who may be unfamiliar with multivariable calculus.

Scientific Ethics Declaration

The author declares that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the author.

Acknowledgements or Notes

This article was presented as an oral presentation at the International Conference on Basic Sciences, Engineering and Technology (www.icbaset.net) conference held in Istanbul/Turkey on August 25-28, 2022.

No funding was received for conducting this study.

References

Avvakumov, S. N., Kiselev, Yu. N., Orlov, M. V., & Taras'ev, A. M. (2010). Profit maximization problem for cobb-douglas and CES production function. *Computational Mathematics and Modeling*, *21*, 336-378.

Angeli, E., Wagner, J., Lawrick, E., Moore, K., Anderson, M., Soderland, L., & Brizee, A. (2010, May 5). General format. http://owl.english.purdue.edu/owl/resource/560/01/ https://doi.org/10.1007/s10598-010-9075-5

Boyd, S., Kim, S.-J., Vanderberghe, L., & Hassibi, A. (2007). A tutorial on geometric programming. *Optimization and Engineering*, *8*, 67-127. https://doi.org/10.1007/s11081-007-9001-7

Beightler, C. S., & Philips, D. T. (1976). Applied geometric programming. New York: John Wiley & Sons.

Bullen, P. S. (2003). *Handbook of means and their inequalities*. Springer Dordrecht. https://doi.org/10.1007/978-94-017-0399-4

Duffin, R. J., Peterson, E. L., & Zener, C. (1967). *Geometric programming – theory and applications*. New York: John Wiley & Sons.

Jehle, G. A., & Reny, P. J. (2011). *Advanced microeconomic theory*. New Jersey: FT Prentice.

Kojić, V. & Lukač, Z. (2018). Solving profit maximization problem in case of the Cobb-Douglas production function via weighted AG inequality and geometric programming. Proceedings from: *2018 IEEE International Conference on Industrial Engineering and Engineering Management (IEEM)*, Bangkok, Thailand, 1900-1903. https://doi.org/10.1109/IEEM.2018.8607446

Liu, S. T. (2006). A geometric programming approach to profit maximization. *Applied Mathematics and Computation*, *182*, 1093-1097. https://doi.org/10.1016/j.amc.2006.04.061

Reklaitis G., Salas, V., & Whinston, A. (1975). *Duality and geometric programming*. https://pure.iiasa.ac.at/id/eprint/283/1/WP-75-145.pdf

Zevelev, A. A. (2014, November). *Closed form solutions in economics*. https://ssrn.com/abstract=2354226

Author Information

Vedran Kojić
Faculty of Economics and Business, University of Zagreb
Trg J. F. Kennedyja 6, 10000 Zagreb, Croatia
Contact e-mail: *vkojic@efzg.hr*

To cite this article:

Kojić, V. (2022). On a geometric programming approach to profit maximization: the case of CES technology. *The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM), 18,* 7-15.