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Finite Element Method for Analysis of Off-Center Connected Continuous Beams

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Abstract: The study of the structural elements goes through the premise of their centric support and centric connection. In reality, it is common to see elements with different cross-sections connected by offset axes, for example to align some of their edges, or those that are supported with off-center supports. Such inter-element eccentricities and off-center supports leads to a change in the stress-strain state from the traditional ones known from the available literature and university textbooks. In previous publications with participation of the author, exact nonlinear and approximate analytic solutions of similar problems for beams under various loading, support and connection conditions have already been published. The finite element method provides an approximate approach for computer analysis. In this paper, results obtained through a computer program created in the MATLAB, based on a finite element accounting for eccentricity in connection and off-centering of support, are demonstrated. The results shows values in the diagrams of internal forces differing by up to 15-30% compared to the classical theory of centric connection and centric supporting.

Keywords: Continuous beam, Different cross-sections, Inter-element eccentricities, Off-center supports, Finite element method

Introduction

In construction practice, it is often assumed that beams and systems of bodies are connected under the premise of their axial support and axial connection. In real structure, however, there is often a divergence from this assumption. The presence of different operating conditions of the structural elements causes the emergence of a more special stressed and strained state. This is proven in a number of studies with the participation of the author (Mladenov & Doicheva, 2011; Doicheva & Mladenov, 2009). Inferences were made in both, linear and non-linear analysis. The latter case is based on the exact differential equation of the deformed beam axis

$\frac{d^2\varphi_i}{ds_i^2}+\frac{F_i}{EI}\left(1-\frac{F_i}{EA}\cos\varphi_i\right)\sin\varphi_i=0$ and the elastic analogy theorem (Mladenov, 1985-86).

The practical application of the obtained results requires an approach applicable to modern computing technology. Over the past few decades, the finite element method (Bankov & Pavlova 1989; Gallagher, 1984; Bhatti, 2005) has proven to be a reliable means of solving complex stress and strain states. In this paper FEM will be used to study continuous beams without and with joints. However, a corresponding finite element is required for this purpose. A stiffness matrix of such a beam-type finite element was obtained by the force method in (Mladenov et al., 2011-2012). Another so-called direct approach will be applied here (Gallagher, 1984; Gavin, 2011).

Method

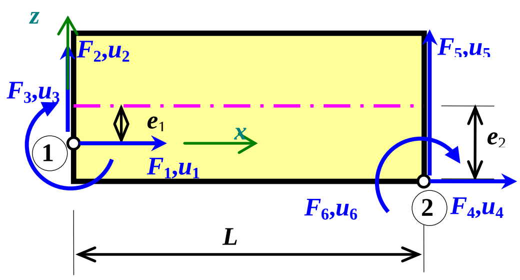

Figure 1. Finite element for unsymmetrical support with nodal forces and displacements

Figure 1 presents the type of finite element with length L, eccentricities e_1 and e_2, respectively, at nodes *1* and *2*, nodal forces, nodal moments as well as unknown displacements and unknown rotations. From now on, for brevity, we will only talk about nodal forces and displacements in a generalized sense. The element is subjected to special bending and normal force. The physico-mechanical characteristics of the cross-section are: E - modulus of elasticity, I - principal moment of inertia, A - area of the cross-section and $i=\sqrt{I/A}$ - radii of gyration.

Derivation of Stiffness Matrix by Direct Approach

The fields of displacements (Gallagher, 1984) $u(x)=a_1x+a_2$; $w(x)=a_3x^3+a_4x^2+a_5x+a_6$ with $w'(x)=3a_3x^2+2a_4x+a_5$ have values for the endpoints respectively:

$$u(0)=a_2;u(L)=a_1L+a_2;\ w'(0)=a_5;u_3\approx -w'(0);\ w'(L)=3a_3L^2+2a_4L+a_5;u_6\approx -w'(L) \tag{1}$$

On the other hand (Figure 1),

$u(0)=u_1+e_1u_3=u_1-e_1w'(0)=u_1-e_1a_5$, and $u(L)=u_4+e_2u_6=u_4-e_2w'(L)=u_4-e_2\left(3a_3L^2+2a_4L+a_5\right)$.

From here follows the relation $\{\mathbf{u}\}=[\mathbf{B}]\{\mathbf{a}\}$; $\mathbf{det(B)}=-L^5$, from where for the coefficients of the displacement functions we can write $\{\mathbf{a}\}=\left[\mathbf{B}^{-1}\right]\{\mathbf{u}\}$, therefore:

$$\begin{Bmatrix} a_1\\ a_2\\ a_3\\ a_4\\ a_5\\ a_6 \end{Bmatrix}=\begin{bmatrix} -1/L & 1/L & 0 & -e_1/L & 0 & e_2/L\\ 1 & 0 & 0 & e_1 & 0 & 0\\ 0 & 0 & 2/L^3 & -1/L^2 & -2/L^3 & -1/L^2\\ 0 & 0 & -3/L^2 & 2/L & 3/L^2 & 1/L\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1\\ u_4\\ u_2=(w_1)\\ u_3=(\alpha_1)\\ u_5=(w_2)\\ u_6=(\alpha_2) \end{Bmatrix} \tag{2}$$

Thus, the longitudinal displacement as a function of the nodal displacements is $u(x)=\left(u_4-u_1+e_2u_6-e_1u_3\right)x/L+u_1+e_1u_3$, and for the axial linear strain is written $\varepsilon_x=u'(x)=\left(u_4-u_1+e_2u_6-e_1u_3\right)/L$. Using Hooke's law, the normal stress and hence the normal force in the beam is determined:

$$\mathsf{N}=\sigma_xA=EA\varepsilon_x=\frac{EA}{L}\lfloor -1\quad 1\quad 0\quad -e_1\quad 0\quad e_2\rfloor\{\mathbf{u}\} \tag{3}$$

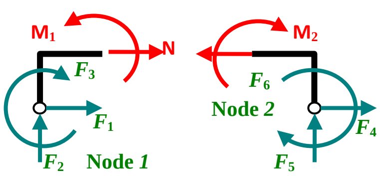

Figure 2. Internal forces at the finite element boundary

On the other hand, the relationship between the normal force and the nodal forces is (see figure 2) $F_{1/4} = \mp N$, so:

$$\begin{Bmatrix} F_1 \\ F_4 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 & e_1 & 0 & -e_2 \\ -1 & 1 & 0 & -e_1 & 0 & e_2 \end{bmatrix}\{\mathbf{u}\} \tag{4}$$

The following coefficients a_3, a_4, a_5, a_6 are expressed by (2)

$$a_3 = \frac{2u_2}{L^3} - \frac{u_3}{L^2} - \frac{2u_5}{L^3} - \frac{u_6}{L^2}; \quad a_4 = -\frac{3u_2}{L^2} + \frac{2u_3}{L^2} + \frac{3u_5}{L^2} + \frac{u_6}{L^2}; \quad a_5 = -u_3; \quad a_6 = u_2 . \tag{5}$$

Substituting (5) into the transverse displacement function yields:

$$w(x) = \left(2\frac{x^3}{L^3} - 3\frac{x^2}{L^2} + 1\right)u_2 + x\left(2\frac{x}{L} - \frac{x^2}{L^2} - 1\right)u_3 + \left(3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}\right)u_5 + x\left(\frac{x}{L} - \frac{x^2}{L^2}\right)u_6 . \tag{6}$$

This result is recorded in matrix form $w(x) = \lfloor \mathbf{N} \rfloor \{\Delta\} = \lfloor N_1\ N_2\ N_3\ N_4 \rfloor \{\Delta\}$, where $\{\Delta\} = \lfloor u_2\ u_3\ u_5\ u_6 \rfloor^T$. N_i are called interpolation functions or shape functions (Gallagher, 1984).

The dependence between the bending moments defined in the end sections is used as follows: $M_1 = -F_1 e_1 + F_3$; $M_2 = F_4 e_2 - F_6$ and the derivatives $w''(0)$ and $w''(L)$, bearing in mind that $M(x) = EIw''(x)$, a system of two linear equations with four unknowns is obtained. Since F_1 and F_4 are already defined, from the above expressions we extract F_3 and F_6, i.e.

$$\begin{Bmatrix} F_3 \\ F_6 \end{Bmatrix} = \frac{EI}{L^3 i^2}\begin{bmatrix} L^2 e_1 & -L^2 e_1 & -6Li^2 & L^2\left(4i^2 + e_1^2\right) & 6Li^2 & L^2\left(2i^2 - e_1 e_2\right) \\ -L^2 e_2 & L^2 e_2 & -6Li^2 & L^2\left(2i^2 - e_1 e_2\right) & 6Li^2 & L^2\left(4i^2 + e_2^2\right) \end{bmatrix}\{\mathbf{u}\}. \tag{7}$$

Finally, through the expressions for F_1, F_3, F_4, F_6 and the conditions for equilibrium of the forces is found:

$$\begin{Bmatrix} F_2 \\ F_5 \end{Bmatrix} = \frac{EI}{L^3 i^2}\begin{bmatrix} 0 & 0 & 12i^2 & -6Li^2 & -12i^2 & -6Li^2 \\ 0 & 0 & -12i^2 & 6Li^2 & 12i^2 & 6Li^2 \end{bmatrix}\{\mathbf{u}\}. \tag{8}$$

Written together (4, 7, 8) give the stiffness matrix of the finite element considering the influence of only the bending moment and the normal force:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \frac{1}{L^3 i^2}\begin{bmatrix} L^2 & 0 & L^2 e_1 & -L^2 & 0 & -L^2 e_2 \\ 0 & 12i^2 & -6Li^2 & 0 & -12i^2 & -6Li^2 \\ L^2 e_1 & -6Li^2 & L^2\left(4i^2 + e_1^2\right) & -L^2 e_1 & 6Li^2 & L^2\left(2i^2 - e_1 e_2\right) \\ -L^2 & 0 & -L^2 e_1 & L^2 & 0 & L^2 e_2 \\ 0 & -12i^2 & 6Li^2 & 0 & 12i^2 & 6Li^2 \\ -L^2 e_2 & -6Li^2 & L^2\left(2i^2 - e_1 e_2\right) & L^2 e_2 & 6Li^2 & L^2\left(4i^2 + e_2^2\right) \end{bmatrix}\begin{Bmatrix} EIu_1 \\ EIu_2 \\ EIu_3 \\ EIu_4 \\ EIu_5 \\ EIu_6 \end{Bmatrix}. \tag{9}$$

Geometric Stiffness Matrix

The element with pre-deformation length dx due to bending only is elongated to dl (Figure 3). Furthermore

$$dl = \sqrt{(dx)^2 + (dw)^2} \approx \left(1 + w'^2/2\right)dx . \tag{10}$$

Therefore, the absolute extension of the element with length dx is $dl - dx = \left(w'^2/2\right)dx$.

The differentially small work of the normal force from this extension will be $d\pi = \mathrm{N}(x)\,(w'^2 dx)/2$, and for the entire beam will be $\Pi_G = \int_0^L (\mathrm{N}(x) w'^2) dx / 2$, which is exactly the potential energy of deformation.

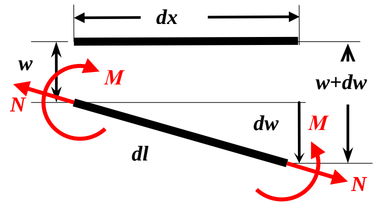

Figure 3. Infinitesimal length deformation

At a constant normal force $\mathrm{N} = N$ we will have $\left(N\int_0^L w'^2 dx\right)/2$.

$$\left(\overline{\Pi}_G = \frac{\Pi_G}{N}\right) = \frac{1}{2}\int_0^L w'^2 dx = \frac{\left(6u_2^2 - 12u_2u_5 + 6u_5^2\right)}{10L} - \frac{(u_2 - u_5)(u_3 + u_6)}{10} + L\frac{2u_3^2 - u_3u_6 + 2u_6^2}{30}. \tag{11}$$

From Castigliano's inverse theorem, we obtain the forces F_{iG}, and the geometric stiffness matrix, which is due only to the influence of a constant normal force and the bending deformation, takes the known form (Gallagher, 1984; Gavin, 2011):

$$\begin{Bmatrix} F_{1G} \\ F_{2G} \\ F_{3G} \\ F_{4G} \\ F_{5G} \\ F_{6G} \end{Bmatrix} = N \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6/5L & -1/10 & 0 & -6/5L & -1/10 \\ 0 & -1/10 & 2L/15 & 0 & 1/10 & -L/30 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6/5L & 1/10 & 0 & 6/5L & 1/10 \\ 0 & -1/10 & -L/30 & 0 & 1/10 & 2L/15 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{N}{EI} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6/5L & -1/10 & 0 & -6/5L & -1/10 \\ 0 & -1/10 & 2L/15 & 0 & 1/10 & -L/30 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6/5L & 1/10 & 0 & 6/5L & 1/10 \\ 0 & -1/10 & -L/30 & 0 & 1/10 & 2L/15 \end{bmatrix} \begin{Bmatrix} EIu_1 \\ EIu_2 \\ EIu_3 \\ EIu_4 \\ EIu_5 \\ EIu_6 \end{Bmatrix}. \tag{12}$$

The use of the geometric stiffness matrix in addition to the basic one turns out to be very useful in the iterative determination of a critical load in frame systems in the Euler sense (Gallagher, 1984; Saouma, 2002))

Results and Discussion

The calculations were performed with a program running in the MATLAB (MATLAB R2017b) environment.

First Example

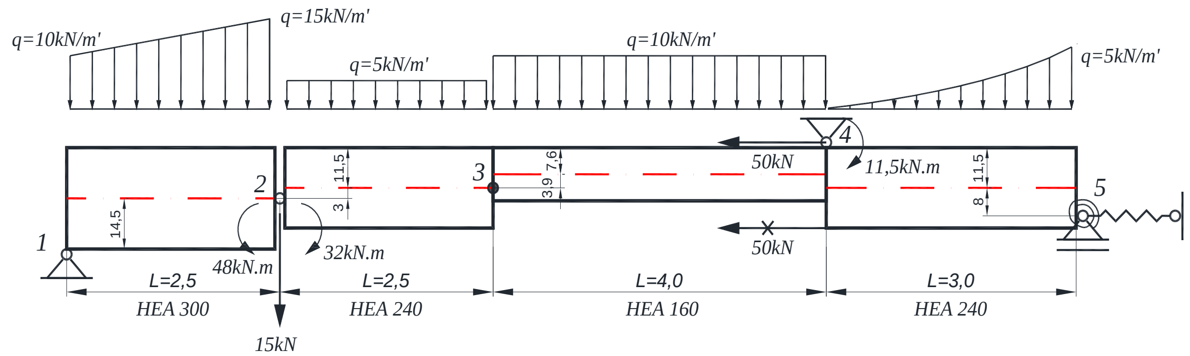

Figure 4. The joint is in the first element axis

Figure 4 shows a beam composed of 4 elements, two of which are with equal cross-section. All sections are aligned along the upper edge of the section, which leads to the divergence of their axes. The connection between the first and second section is with joint. Initially, the joint is in the first element axis. The concentrated forces and moments are introduced at the elements nodes. The stiffnesses of the linear and rotational springs are related to the stiffnesses of the beam. The linear springs is brought to the stiffness of tension (compressure) of the beam by coefficient ζ, as $k_i = \zeta_i \dfrac{EA_i}{L_i}$. The stiffness of the rotational springs is reduced to the bending stiffness of the beam by the coefficient γ and is $c_i = \gamma_i \dfrac{EI_i}{L_i}$. In the example considered are accepted $\zeta = 0,25$ and $\gamma = 0,5$.

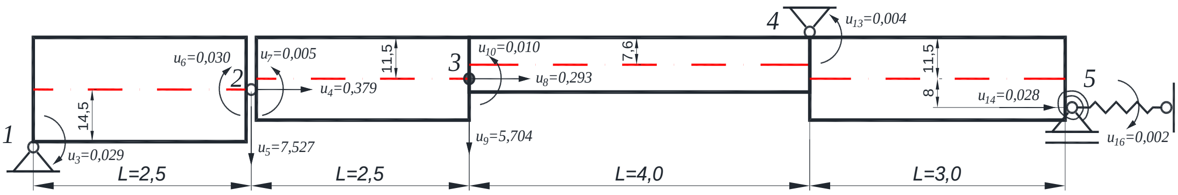

Figure 5. Displacements and rotations in the nodes

The results are shown graphically in Figure 5 and Figure 6.

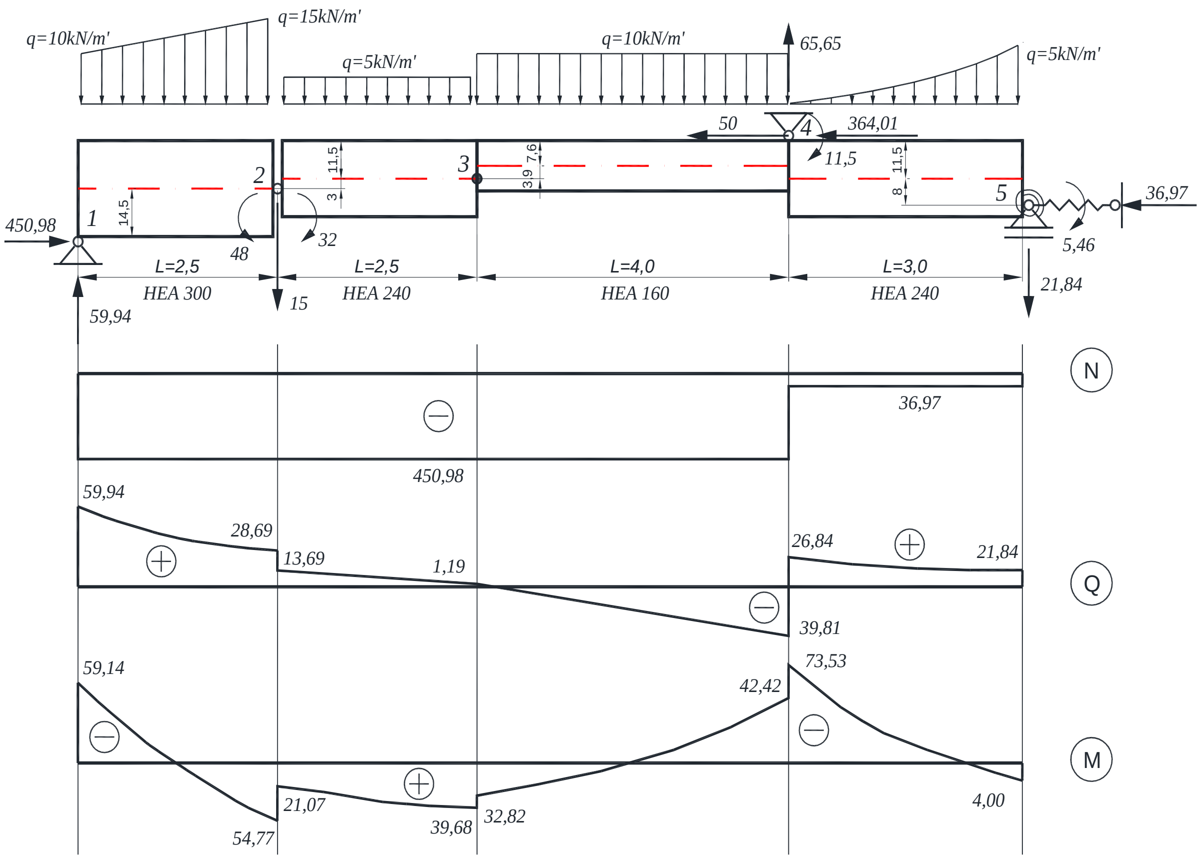

Figure 6. Initial force when the joint is in the first element axis

The second case is when the joint is located along the lower edge of the second element and all other conditions from the first case are preserved, Figure 7.

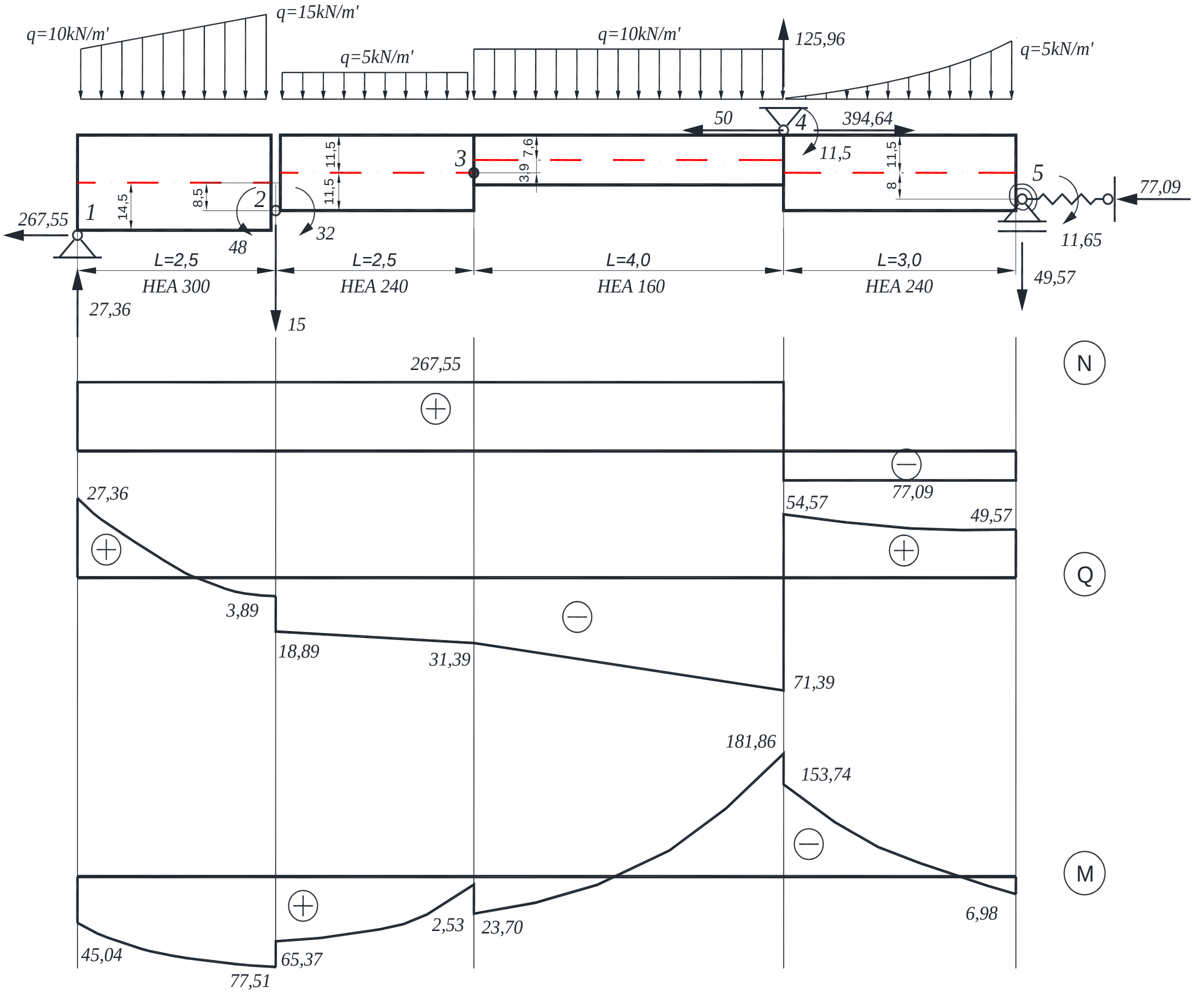

Figure 7. Beam with joint on the lower edge of the second element

In the two cases of joint position, the initial forces have different values. The large axial force is striking and the fact that if in the first case normal force is compressive, then in the second it is tensile.

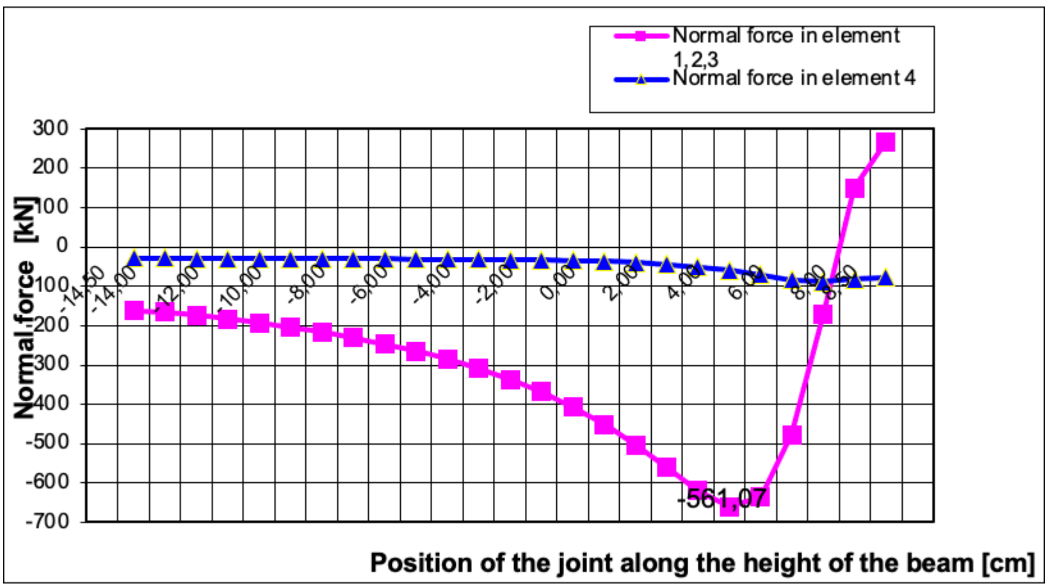

Figure 8. Change in normal force when moving the joint along the height of the beam

The limit of variation of the normal force during the displacement of the joint along the height of the beam is shown in Figure 8. When placing the joint around the upper edge on the beam, the horizontal force is compressive. This force numerically exceeds the vertical transverse loads by about two times and continues to increase as the joint moves towards the axis. The extreme value is slightly below the axis of the first element. Moving the joint to the bottom edge of a second element results in a rapid reset and reversal to the tension direction with a value commensurate with the maximum compressive force. The example is indicative for the importance of research on the different position of intermediate joints in non-axial connection and off-center support of continuous beams.

Second Example

A classic example about the occurrence of a complex stressed and strained state caused by the off-center support along the lower or upper edge is the crane beam (Figure 9).

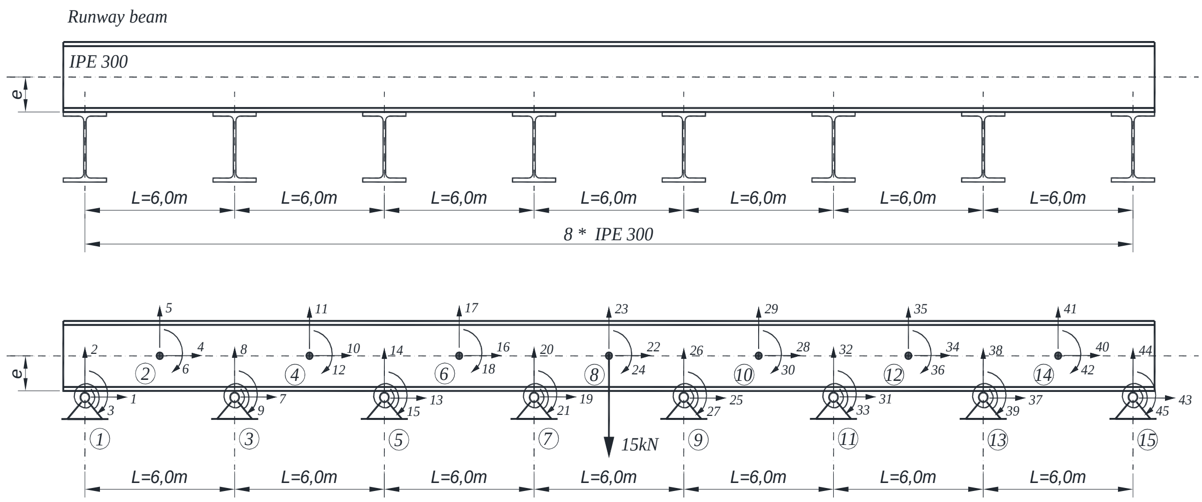

Figure 9. Crane beam

The crane beam is made of an IPE 300 profile. The crane beam is supported on an analogous profiles with a cross section of IPE 300. The bolted connections between the profiles are modeled as fixed supports. There will be a tendency to torsion on the support beams, which will lead to the occurrence of an additional moment accounted by the rotation springs in the supports. Their coefficient $c = \frac{GI_t}{L}$ has the value $c = \frac{8000.10^4.19,77.10^{-8}}{0,20} = 79,08\frac{\text{kN.m}}{\text{rad}}$. On the other hand, the coefficient of the rotation spring reduced to the bending stiffness of the beam is $c = \gamma\frac{EI}{L_i}$, where L_i is the length of the *i*-th finite element to which the stiffness of the spring is reduced. It follows that $\gamma = \frac{cL_i}{EI} = \frac{79,08.3}{2,1.10^8.8356.10^{-8}} = 0,0135$.

The results of the program are shown in Figure 10.

The large horizontal force in the loaded span of the beam, which exceeds the value of the transverse load as well as the jumps into the moment diagram in the area of the supports are striking. They are explained precisely by the displacement of the supports from the axis. If the supports were at the axis, as is commonly assumed, then the moment diagram values would be as shown by the dashed line. They differ from the real ones by 14.5% in the span of the beam and 27% in the area above the support. There is a case where, if the support is in the axis, like the classical cases, the beam will be dimensioned as a special bending. The presence of a horizontal force, however, requires that it be considered as a bending combined with tension or compression.

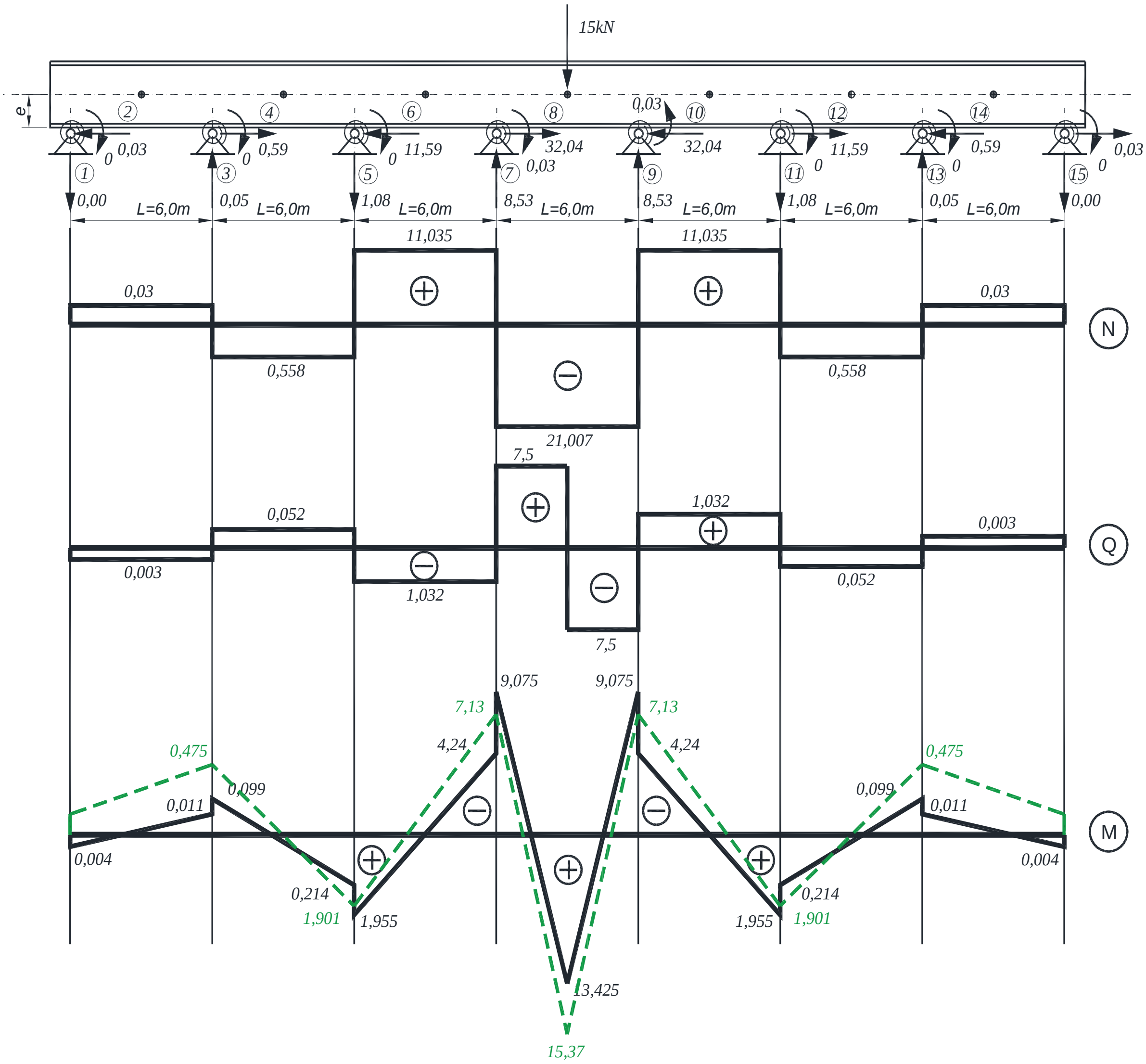

Figure 10. Numerical results for crane beam

Conclusion

The stiffness matrix for off-center connection of the finite elements is derived.

Examples of beams with off-center support and eccentrical connection are considered.

The influence on the internal forces at the different positions of the connecting joint between two elements of a continuous beam is shown.

The values of the normal forces, which many times exceed the vertical loads are demonstrated.

An example of a crane beam shows the effect of off-center supports on the stress and strain state. Large horizontal forces are observed. Bending moments over some supports and in the some span of the beam exceed those obtained when the beam is axially supported between 14,5-27%

Scientific Ethics Declaration

The author declares that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the author.

Notes

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