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Approximate Analytic Solution of Riccati Equation with Fractional Order of Multi-Parameters

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Abstract: In this paper, we present an approximate analytic solution of the Riccati equation with fractional order of multi-parameters. The fractional order of Caputo types with generalized Mittag–Leffler kernel is adaptive, this kind of fractional derivative has three fractional parameters. Several properties of the fractional derivative and integral are studied. We use the homotopy analysis method to generate the approximate analytic solution to the problem. The effect of the fractional parameters on the behavior of the solution is studied, each parameter of the fractional derivative can change not only the solution behaviors but also the existence of the solution. Two examples are presented to demonstrate the efficiency of the method. Comparisons of the exact solution and the approximate solution in the case of the standard derivative are made. For the fractional case, we calculate the residual error of the approximate solution. In all cases, the solution is accurate and simply applies.

Keywords: Fractional calculus, Riccati equation, Homotopy analysis method.

Introduction

Fractional calculus becomes one of the most interesting subjects in the area of applied mathematics. Several definitions of the fractional derivatives were introduced in terms of the local and the memory of the functions. Applications of fractional calculus appear in the differential equations which almost describe a real-live phenomenon.

The Riccati equation is utilized in many branches of mathematics, including physics, algebraic geometry, and conformal mapping theory. It also shows up in a lot of practical issues. The Count Jacopo Francesco Riccati of Italy is honored by having his name attached to the Riccati differential equation (RDE) (1676-1754). The foundational theories of the Riccati equation are covered in the book by Reid (Reid, 1972), which also includes applications to random processes, optimal control, and diffusion issues. A well-known nonlinear differential equation, the Riccati equation has numerous uses in the fields of engineering and science, including resilient stabilization, stochastic realization theory, network synthesis, optimal control, and financial mathematics.

The Riccati differential equation (RDE) of fractional order has been explored by numerous authors; for instance, (Momani & Shawagfeh, 2006), the authors created the Adomain decomposition method for the solution of RDE of fractional order. Some analytical methods for the resolution of RDE are provided (Pala & Ertas, 2017). The authors (Biazar & Eslami, 2010) used the differential transform approach to arrive at the RDE solution. The author (Tsai, 2010) created a Laplace-transform Adomain decomposition technique for the resolution of (RDE). An analytical approach based on the homotopy analysis method (HAM) is suggested to resolve nonlinear (RDE) with fractional order (Cang et. al., 2009). Using the homotopy approach, Odibat and Momani devised an algorithm for the quadratic Riccati differential equation of fractional order (Odibat & Momani, 2008).
Recently, Abdeljawad (Abdeljawad, 2019) introduced a generalized Atangana-Baleanu Caputo (GABC) fractional derivative based on the Mettag-Lefller function kernel which contains three parameters. Srivastava et.

al. investigated the definition with Legendre polynomials for solving several physical models (Srivastava et. al., 2021). Moreover, the fractional parabolic differential equation under the GABC fractional derivative is solved by the HAM (Alomari et. al., 2020). To the best of our knowledge, this paper will introduce the solution of RDE using GABC for the first time. The Riccati differential equation in fractional case using generalized Atangana-Baleanu Caputo (GABC) definition can be written as:

$$0^{ABC}D_t^{\alpha,\mu,1}y(t) = p(t) + q(t)y(t) + r(t)y^2(t), \tag{1}$$

subject to $y(0) = a$.

The following results for the GABC fractional derivative can be found in (Abdeljawad, 2019).

Definition 1 *The generalized Atangana-Baleanu Caputo (GABC) fractional derivative with Mittag-Leffler kernel of three parameters* $E^{\gamma}_{\alpha,\mu}(\lambda, t)$, *is defined by*

$$(a^{ABC}D^{\alpha,\mu,\gamma}f)(x) = \frac{M(\alpha)}{1-\alpha}\int_a^x E^{\gamma}_{\alpha,\mu}(\lambda, x-t)f'(t)dt, \tag{2}$$

where $0 < \alpha < 1$, $Re(\mu) > 0$, $\gamma \in \mathbb{R}$, $\lambda = \frac{-\alpha}{1-\alpha}$, $E^{\gamma}_{\alpha,\mu}(\lambda, z) = \sum_{k=0}^{\infty}\frac{\lambda^k(\gamma)_k z^{\alpha k+\mu-1}}{k!\Gamma(\alpha k+\mu)}$, *and* $(\gamma)_k = \gamma(\gamma+1)\cdots(\gamma+k-1)$ *is the Pochhamme function.*

For $\gamma = 1,2,3,\cdots$, the AB fractional integrals of order $0 < \alpha, \mu \leq 1$ can be written as

$$(a^{AB}I^{\alpha,\mu,\gamma}f)(x) = \sum_{i=0}^{\gamma}\binom{\gamma}{i}\frac{\alpha^i}{M(\alpha)(1-\alpha)^{i-1}}(_a I^{\alpha i+1-\mu}f(x).$$

Theorem 2 *For* $0 < \alpha < 1, \mu > 0, \gamma \in \mathbb{N}$, *and* $\lambda = \frac{-\alpha}{1-\alpha}$, *we have*

$$\begin{aligned}(a^{AB}I^{\alpha,\mu,\gamma}a^{ABC}D^{\alpha,\mu,\gamma}f)(x) &= f(x) - f(a)\sum_{k=0}^{\gamma}(-1)^k\lambda^k E^{\gamma}_{\alpha,\alpha k+1}(\lambda, x-a)\\ &= f(x) - f(a).\end{aligned} \tag{3}$$

HAM Solution

The homotopy analysis method provides an approximate analytical solution for various nonlinear problems. In this chapter, we extend the applications of the homotopy analysis method to the general form of the time-fractional partial differential equation:

$$0^{ABC}D^{\alpha,\mu,\gamma}\eta(\xi) = N[u(t)], \qquad 0 < \alpha \leq 1, \mu > 0, \tag{4}$$

subject to the initial condition: $\eta(0) = a$, where N is a non-linear operator, ξ denotes an independent variable, and $\eta(\xi)$ is an unknown function. Firstly, we construct the homotopy map (Alomari et. al. 2020):

$$(1-q)L[\phi(\eta(\xi);q) - \eta_0(\xi)] = \hbar q(0^{ABC}D^{\alpha,\mu,\gamma}\phi(\eta(\xi);q) - N[\phi(\eta(\xi);q)]), \tag{5}$$

where $q \in [0,1]$ is an embedding parameter, $\hbar$ is a nonzero convergent control parameter, L is an auxiliary linear operator, $\eta_0(\xi)$ denotes an initial approximation of the solution, and $\phi(\eta(\xi);q)$ is an unknown function. When $q = 0$ and $q = 1$, it holds

$$\phi(\eta(\xi);0) = \eta_0(\xi), \qquad \phi(\eta(\xi);1) = \eta(\xi). \tag{6}$$

Thus, as q increases from 0 to 1, $\phi(\eta(\xi);q)$ varies from the initial guess $\phi(\eta(\xi);0)$ to the exact solution $\phi(\eta(\xi);1)$. For succinctness, equation (5) is called the zero-order deformation equation.

According to HAM, we have the freedom to choose the auxiliary parameter $\hbar$, the initial approximation $\eta_0(\xi)$, and the auxiliary linear operator L. we can assume that all of them are properly chosen so that the solution $\phi(\eta(\xi);q)$ of the zero-order deformation equation (5) exists for $0 \leq q \leq 1$, and besides, the ith-order deformation derivative. Define

$$\eta_i(\xi) = \frac{1}{i!}\frac{\partial^i\phi(\eta(\xi);q)}{\partial q^i}|_{q=0}. \tag{7}$$

Expanding, $\phi(\eta(\xi);q)$ in Taylor's series with respect to q, we have

$$\phi(\eta(\xi);q) = \phi(\eta(\xi);0) + \sum_{i=1}^{\infty} \frac{1}{i!}\frac{\partial^i \phi(\eta(\xi);q))}{\partial q^i}|_{q=0} q^i. \tag{8}$$

From equations (6) and (7), the above power series can be written as:

$$\phi(\eta(\xi);q) = \eta_0(\xi) + \sum_{i=1}^{\infty} \eta_i(\xi) q^i. \tag{9}$$

Substitute the value of $\phi(\eta(\xi);q)$ into equation (5), and we get

$$(1-q)L[\sum_{i=1}^{\infty} \eta_i q^i] = \hbar(0^{ABC}D^{\alpha,\mu,\gamma} \sum_{i=0}^{\infty} \eta_i q^{i+1} - qN[\sum_{i=0}^{\infty} \eta_i q^i]). \tag{10}$$

By equating like powers of q from both sides in Eq.(10), we get

$$q^1: L[\eta_1(\xi) - 0] = \hbar(0^{ABC}D^{\alpha,\mu,\gamma} \eta_0(\xi) - R_1),$$

$$q^2: L[\eta_2(\xi) - \eta_1(\xi)] = \hbar(0^{ABC}D^{\alpha,\mu,\gamma} \eta_1(\xi) - R_2),$$

$$\vdots$$

$$q^n: L[\eta_n(\xi) - \eta_{n-1}(\xi)] = \hbar(0^{ABC}D^{\alpha,\mu,\gamma} \eta_{n-1}(\xi) - R_n),$$

where

$$R_n = \frac{1}{(n-1)!}\frac{\partial^{n-1} N[\Phi(\eta(\xi),q)]}{\partial q^{n-1}}|_{q=0}. \tag{11}$$

The initial conditions define as $\Phi(\eta(0);q) = \eta_0(0) + \sum_{i=1}^{\infty} \eta_i(0) q^i = a$. Thus $\eta_0(0) = a$ and $\eta_i(0) = 0$, where $i = 1,2,3,\cdots$. Assume that the auxiliary linear operator L, the initial guess $\eta_0(t)$, and the auxiliary parameter $\hbar$ is selected such that the series (9) is convergent at $q = 1$, then due to (6) we have

$$\eta(\xi) = \eta_0(t) + \sum_{i=1}^{\infty} \eta_i(\xi).$$

Additionally, the values of the auxiliary parameter $\hbar$ have a significant impact on the convergence and rate of approximation for the HAM solution. It is simple to select an appropriate value for $\hbar$ that will guarantee that the solution series is convergent. Finding the valid region of $\hbar$, which relates to line segments almost parallel to the horizontal axis, is simple. This indicates that during this region, the solution is independent of $\hbar$. Therefore, by selecting an appropriate value for this auxiliary parameter, the convergence region and pace of the solution series can be significantly increased. To get the ideal value of $\hbar$, we first fixed $\alpha = \mu$ and used the least square approach. Now, consider the residual error.

$$Res(\xi) = 0^{ABC}D^{\alpha,\mu,\gamma}\eta(\xi) - N[\eta(\xi)], \tag{12}$$

and the average residual error function

$$\zeta(\hbar) = \frac{1}{(N_1+1)}\sum_{j=0}^{N_1} Res^2(\xi_i), \tag{13}$$

where $\xi_j = \frac{jK}{N_1}$. Therefore, we will use the averaged residual error (13) to find the optimal values of the unknown convergence-control parameter $\hbar$. Note that $\zeta(\hbar)$ contains unknown convergence-control parameter $\hbar$. The more quickly $\zeta(\hbar)$ decreases to zero, the faster the corresponding homotopy-series solution converges. So, the optimal values of the convergence-control parameter $\hbar$ are given by the minimum of $\zeta(\hbar)$, corresponding to a set of a nonlinear algebraic equation. $\frac{\partial\zeta(\hbar)}{\partial\hbar} = 0$. Using the symbolic computation software Mathematica, we directly employ the command Minimize to get the optimal convergence-control parameter $\hbar$.

Applications

In this section, we introduce the solution of two examples; linear and nonlinear Riccati fractional differential equation with three parameters.

Example 1 Consider the linear problem:

$$0^{ABC}D_t^{\alpha,\mu,1}y(t) = y(t), \tag{14}$$

subject to the initial condition:

$$y(0) = 1. \tag{15}$$

At $\alpha = \mu = 1$ the exact solution is $y(t) = e^t$ which will be useful for the comparison of different approximations. By choosing $y_0(t) = y(0)$, and the linear operator $L = 0^{ABC}D_t^{\alpha,\mu,1}$, the zero-order of deformation (5) becomes

$$(1-q)0^{ABC}D_t^{\alpha,\mu,1}[\sum_{i=0}^{\infty} y_i(t)q^i - y_0(t)] = \hbar q(0^{ABC}D_t^{\alpha,\mu,1}\sum_{i=0}^{\infty} y_i(t)q^i - \sum_{i=0}^{\infty} y_i(t)q^i), \tag{16}$$

$$0^{ABC}D_t^{\alpha,\mu,1}[\sum_{i=1}^{\infty} y_i(t)q^i - \sum_{i=1}^{\infty} y_i(t)q^{i+1}] = \hbar(0^{ABC}D_t^{\alpha,\mu,1}\sum_{i=0}^{\infty} y_i(t)q^{i+1} - \sum_{i=0}^{\infty} y_i(t)q^{i+1}), \tag{17}$$

where

$$\Phi(y(t),q) = y_0 + \sum_{i=1}^{\infty} y_i q^i.$$

Balancing the coefficients of equal powers of q, we have the following set of infinite linear fractional differential equations:

$$0^{ABC}D_t^{\alpha,\mu,1}[y_n(t) - \chi_n y_{n-1}(t)] = \hbar\left[0^{ABC}D_t^{\alpha,\mu,1}y_{n-1}(t) - y_{n-1}(t)\right],$$

where $\chi_n = \begin{cases} 0, n \le 1 \\ 1, n > 1 \end{cases}$.

For the initial condition, we have

$$\Phi(y(0),q) = y_0(0) + \sum_{i=1}^{\infty} y_i(0)q^i = 1, \qquad \text{thus, } y_0(0) = 1,\ y_i(0) = 0,\ \text{i=1,2,3,.}$$

By applying the integral operator $0^{AB}I^{\alpha,\mu,\gamma}$ on both sides with $\gamma = 1$, and using equation (3), we achieve the general form of the infinite linear fractional differential equations given by

$$y_n(t) = (\chi_n + \hbar)y_{n-1}(t) - (\chi_n + \hbar)y_{n-1}(0) - \hbar 0^{AB}I^{\alpha,\mu,1}[y_{n-1}(t)]. \tag{18}$$

Thus, the N-th order HAM approximate solution is given by

$$y(t) = y_0(t) + \sum_{i=1}^{N} y_i(t). \tag{19}$$

The explicit expression given by (19) contains the auxiliary parameter $\hbar$. This parameter determines the convergence region. Thus the solution depends on the fractional parameters α and μ. We fixed $\mu = 0.5$ and vary $\alpha = 0.2, 0.5, 0.9$. The optimal value of $\hbar$ can be determined by minimizing the $\zeta(\hbar)$ as given in Figure 1. The effect of varying the fractional parameters α and μ the $y(t)$ is presented in Figure 2. Similarly, we plot the residual error $Res(t)$ with different values of fractional derivative in Figure 3.

Table 1 gives the convergent control parameter $\hbar$ and its ARE for several values of α and μ using a 6-order of approximation. Now, we fixed $t = 0$, and calculate the solution for several values of α and μ as in Table 2, for μ= 1 the only case that $y(0) = 1$ happened if $\hbar$ = 0 which is a contradiction with the HAM framework. So, equation (14) has no solution in the case of $\mu = 1$ which means this equation has no solution in the ABC fractional type.

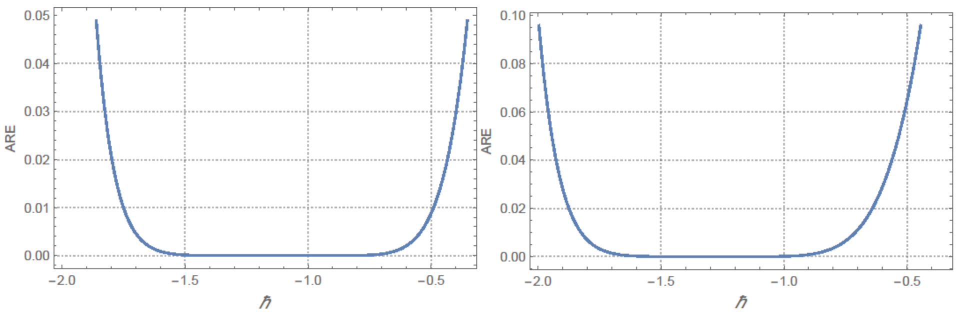


Figure 1. Average residual error with $\hbar$ for α = 0.9 (Left), 0.5(Right).

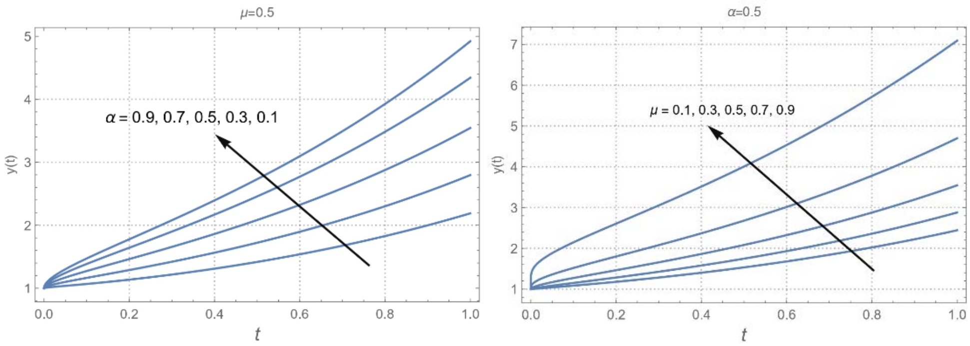


Figure 2. y(t) for example 1 with different values of α (Left), and μ (Right).

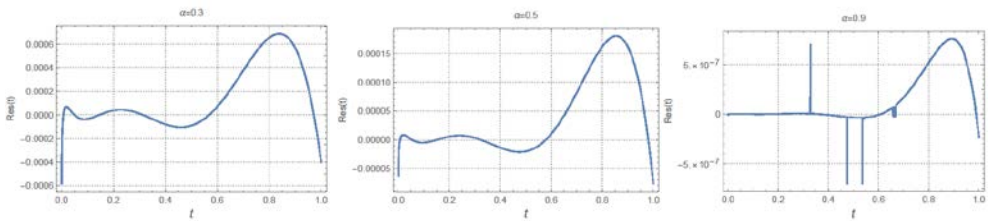


Figure 3. Residual error with optimal $\hbar$ given in table 3.1 for α = 0.3, 0.5, 0.9, respectively.

Table 1. ARE and its optimal $\hbar$ for example 1 at μ=0.5, vary α, and α=0.5, vary μ.

α	ARE	$\hbar$	μ	ARE	$\hbar$
0.1	8.32917×10^{-7}	-1.35798	0.1	3.08439×10^{-12}	-1.0527
0.3	1.28684×10^{-7}	-1.2891	0.3	1.72733×10^{-10}	-1.10178
0.5	6.33778×10^{-9}	-1.19928	0.5	6.33778×10^{-9}	-1.19928
0.7	8.01290×10^{-11}	-1.11493	0.7	1.48121×10^{-6}	-1.38789
0.9	8.36700×10^{-14}	-1.04322	0.9	0.00128426	-1.64569

Table 2. α, μ, and initial condition for example 1

α	μ	Initial condition
0.9	1	$1. - 0.1\,\hbar + 0.01\,\hbar^2 - 0.001\,\hbar^3 + 0.0001\,\hbar^4 - 0.00001\,\hbar^5 + 1.* 10^{-6}\hbar^6$
	0.9, 0.5	1
0.5	1	$1. -0.5\hbar + 0.25\hbar^2 - 0.125\hbar^3 + 0.0625\hbar^4 - 0.03125\hbar^5 + 0.015625\hbar^6$
	0.9, 0.5	1

Example 2 Consider the following ABC fractional Riccati equation:

$$ {}_{0}^{ABC}D^{\alpha,\mu,1}y(t) + y(t) - y^2(t) = 0, \tag{20}$$

with the initial condition,

$$ y(0) = 0.5. \tag{21}$$

The exact solution in the standard case is

$$y(t) = \frac{e^{-t}}{e^{-t}+1}. \tag{22}$$

The homotopy expression for (20) will be,

$$ {}_{0}^{ABC}D^{\alpha,\mu,1}[y_n(t) - \chi_n y_{n-1}(t)] = h[{}_{0}^{ABC}D^{\alpha,\mu,1} y_{n-1}(t) + y_{n-1}(t) - \sum_{j=0}^{n-1} y_{n-1-j}(t) y_j(t)], \tag{23}$$

for $n = 1,2,3,\cdots$, we choose the initial guess $y_0(t) = 0.5$, then applying $0^{AB}I^{\alpha,\mu,1}$ In the above equation, the n-order can be written as:

$$y_n(t) = (\chi_n + \hbar) y_{n-1}(t) - (\chi_n + \hbar) y_{n-1}(0) + \hbar[0^{AB}I^{\alpha,\mu,1}[y_{n-1}(t) - \sum_{j=0}^{n-1} y_{n-1-j}(t) y_j(t)]]. \tag{24}$$

Applying analysis steps, we find the other N-terms. In Figure 4, we plot the behavior of the solution by varying the new fractional parameters μ and α. The residual error of the solutions for several values of α is plotted in Figure 5. Table 2 gives the convergent control parameter and it's ARE for several values of α and μ using 6-order of approximation. We observed that if μ = 1 and varies $0 < \alpha < 1$ (standard ABC derivative) the initial condition will not satisfy (i.e $y(0) = 0.5 + 0.025\,\hbar - 0.0000625\,\hbar^3 + 3.125 \times 10^{-7}\hbar^5$and ($\hbar \neq 0$)); which means that it may have no solution.

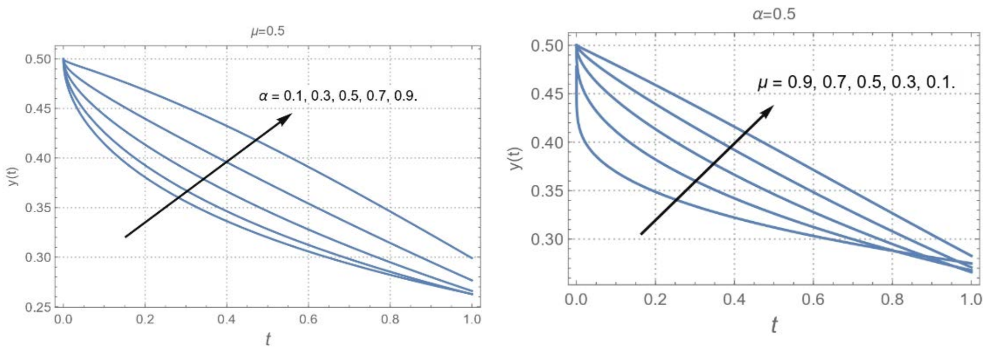
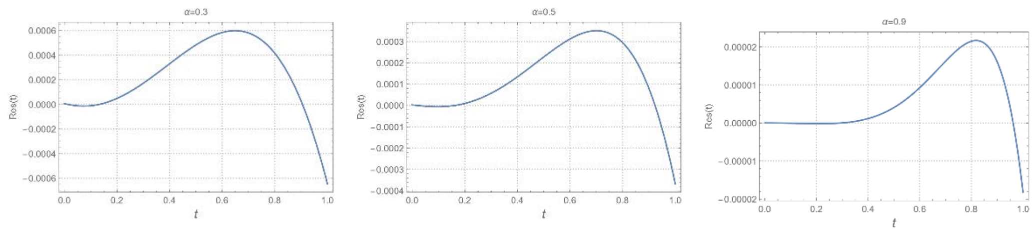


Figure 4. y(t) for example 2 with different values of α (Left), and μ (Right).

Figure 

5. Residual error with optimal $\hbar$ given in table 2 for α = 0.3, 0.5, 0.9 respectively.

Table 3. ARE and its optimal $\hbar$ for example 2 with different values of α with μ=0.5, and different values of μ with α=0.5.

α	ARE	$\hbar$	μ	ARE	$\hbar$
0.1	2.43308×10^{-7}	-0.814443	0.1	9.29851×10^{-10}	-0.911501
0.3	1.49506×10^{-7}	-0.826851	0.3	8.28239×10^{-9}	-0.882866
0.5	4.73852×10^{-8}	-0.850767	0.5	4.73852×10^{-8}	-0.850767
0.7	5.75513×10^{-9}	-0.88488	0.7	1.66378×10^{-7}	-0.81843
0.9	1.28970×10^{-10}	-0.927909	0.9	3.39087×10^{-7}	-0.790958

Conclusion

In this study, we implemented fractional integrals of any order and the fractional operator Caputo type (ABC) with Mittag Leffler kernels in three parameters to analyze the Riccati equation using the homotopy analysis method. Approximate solutions to linear and nonlinear fractional differential equations are calculated using this method. Unlike all other analytic methods, it allows us to easily adjust and control the convergence region of the series solution. The accuracy of the approximate solutions was validated by computing the solution's residual error. The employed method is used to analyze and solve the well-known fractional Riccati equation, which is based on rapidly convergent series with easily compatible components. With a few terms, the HAM is effective and reveals the existence of the solution, and the amount of error is small. We recommend this method for dealing with such issues. The method is straightforward, and it is the first approach to dealing with such issues.

Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

Acknowledgements or Notes

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