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General Upper Bounds for the Numerical Radii of Powers of Hilbert Space Operators

Mohammed AL-DOLAT
Jordan University of Science and Technology

Abstract: In this paper, we will present several upper bounds for the numerical radii of a 2×2 operator matrices. We use these bounds to generalize and improve some well-known numerical radius inequalities. We provide a refinement of an earlier numerical radius inequality due to (Bani-Domi & Kittaneh, 2021) [Norm and numerical radius inequalities for Hilbert space operators], (Bani-Domi & Kittaneh, 2021) [Refined and generalized numerical radius inequalities for 2×2 operator matrices] and (Al-Dolat & Kittaneh, 2023) [Upper bounds for the numerical radii of powers of Hilbert space operators].

Keywords: Numerical radius, Usual operator norm, Operator matrix, Buzano inequality.

Introduction

Let $\mathcal{B}(\mathcal{H})$ be the C^* $-$algebra of all bounded linear operators on the complex Hilbert space $\mathcal{H}$. Recall that the numerical radius $w(.)$ and the usual operator norm $||.||$ are, respectively, defined by

$$w(Y) = \sup_{||x||=1} |\langle Yx, x\rangle| \text{ and } \|Y\| = \sup_{||x||=1} \|Yx\| \text{ where} Y \in \mathcal{B}(\mathcal{H}).$$

A fundamental relation between the norms $w(.)$ and $||.||$ is the following inequality

$$\frac{1}{2}\|Y\| \le w(Y) \le \|Y\| \text{ for every } Y \in \mathcal{B}(\mathcal{H}). \tag{1.1}$$

Many mathematicians are interested in giving refinements for the inequalities in (1.1). For example, in Kittaneh (2005) Kittaneh provided the following improvement

$$\frac{1}{4}\||Y|^2 + |Y^*|^2\| \le w^2(Y) \le \frac{1}{2}\||Y|^2 + |Y^*|^2\| \text{ for every } Y \in \mathcal{B}(\mathcal{H}) \tag{1.2}$$

where $|Y| = (Y^*Y)^{\frac{1}{2}}$.

In El-Haddad and Kittaneh (2007), the authors showed that the upper bound in (1.2) can be generalized as follows:

$$w^{2r}(Y) \le \frac{1}{2}\||Y|^{2r} + |Y^*|^{2r}\| \text{ forevery r} \ge 1 \text{ and Y} \in \mathcal{B}(\mathcal{H}). \tag{1.3}$$

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Recently, in Al-Dolat and Kittaneh (2023), Al-Dolat and Kittaneh have refined the inequality (1.3) by showing that

$$w^{2r}(Y) \leq \frac{1+\alpha}{4} \left\| |Y|^{2r} + |Y^*|^{2r} \right\| + \frac{1-\alpha}{2} w^r(Y^2) \text{ for every } Y \in \mathcal{B}(\mathcal{H}),\ r \geq 1 \text{ and } \alpha \in [0,1].$$
(1.4)

In Kittaneh (2003), Kittaneh showed the following inequality

$$w(Y) \leq \frac{1}{2} \left\| |Y| + |Y^*| \right\| \quad \text{for every } Y \in \mathcal{B}(\mathcal{H}). \tag{1.5}$$

In Dragomir (2009), Dragomir showed that the numerical radius of a product of two operators has the following upper bound

$$w^r(Y^*X) \leq \frac{1}{2} \left\| |X|^{2r} + |Y|^{2r} \right\| \text{ for every } r \geq 1 \text{ and } X, Y \in \mathcal{B}(\mathcal{H}). \tag{1.6}$$

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{H}^{(2)} = \mathcal{H} \bigoplus \mathcal{H}$ denote the $2-$copies of $\mathcal{H}$. Based on this decomposition every operator $Y \in \mathcal{B}(\mathcal{H}^{(2)})$ has a 2×2 operator matrix representation

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

With $Y_{ij} \in \mathcal{B}(\mathcal{H})$ where $i, j \in \{1,2\}$. To learn more about the numerical radii of operator of matrices and their applications, one can refer to (Al-Dolat et al., 2016 ; Al-Dolat & Jaradat, 2023).

In this paper, we give new upper bounds for the numerical radii of 2×2 operator matrices. Based on those bounds, we obtain refinements of the inequality (1.4). Also, we refine earlier numerical radius inequalities for an operator of matrices obtained in (Bani-Domi & Kittaneh, 2021; Al-Dolat & Kittaneh, 2023).

Results and Discussion

For our purpose, we need to recall a few well-known lemmas.

Lemma 2.1 *(Kittaneh, 1988). Let* $Y \in \mathcal{B}(\mathcal{H})$ *be a positive operator and let* $x \in \mathcal{H}$ *with* $||x|| = 1$. *Then*

$$\langle Yx, x \rangle^r \leq \langle Y^r x, x \rangle \quad \text{for every } r \geq 1.$$

Lemma 2.2 *(Aujla & Silva, 2003). Let* f *be a non-negative convex function on* $[0, \infty)$ *and* $X, Y \in \mathcal{B}(\mathcal{H})$ *be positive operators. Then*

$$\left\| f\left(\frac{X+Y}{2}\right) \right\| \leq \left\| \frac{f(X)+f(Y)}{2} \right\|.$$

In particular,

$$\|(X+Y)^r\| \leq 2^{r-1} \|X^r + Y^r\| \quad \text{for every } r \geq 1.$$

Lemma 2.3 *(Hirzallah & Kittaneh, 2011). Let* $X, Y \in \mathcal{B}(\mathcal{H})$. *Then*

(a)

$$w\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \max\{w(X), w(Y)\},$$

(b)

$$w\left(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\right) = \max\{w(X+Y), w(X-Y)\}.$$

In particular,

$$w\left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}\right) = w(Y).$$

Lemma 2.4 *(Buzano, 1974). Let* $u, v, w \in \mathcal{H}$ *with* $\|w\| = 1$. *Then*

$$|\langle u, w\rangle\langle w, v\rangle| \le \frac{1}{2}(\|u\|\|v\| + |\langle u, v\rangle|).$$

Lemma 2.5 *(Moradi & Sababheh, 2021). Let* $X, Y \in \mathcal{B}(\mathcal{H})$. *be self-adjoint. Then*

$$w^2(X + iY) \le \|X^2 + Y^2\|.$$

Our first main result in this paper provides a refinement for the upper bound given in (Ajula & Silva, 2003, Theorem 2.6).

Theorem 2.6 *Let* $X, Y \in \mathcal{B}(\mathcal{H})$. *Then for every* $\alpha \in [0,1]$ *and* $r \ge 2$, *we have*

$$w^r(Y^*X) \le \frac{\alpha}{2}\||X|^r + |Y|^r\| w^{\frac{r}{2}}(Y^*X) + \frac{1-\alpha}{2} w^2(|X|^r + i|Y|^r).$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. Then by letting $u = Xx$ and $v = Yx$ in Lemma 2.5, we have

$$|\langle Y^*Xx, x\rangle|^r = \alpha|\langle Xx, Yx\rangle|^{\frac{r}{2}}|\langle Xx, Yx\rangle|^{\frac{r}{2}} + (1-\alpha)|\langle Xx, Yx\rangle|^r$$

$$\le \alpha\|Xx\|^{\frac{r}{2}}\|Yx\|^{\frac{r}{2}}|\langle Y^*Xx, x\rangle|^{\frac{r}{2}} + (1-\alpha)\|Xx\|^r\|Yx\|^r$$

$$\le \frac{\alpha}{2}(\|Xx\|^r + \|Yx\|^r)|\langle Y^*Xx, x\rangle|^{\frac{r}{2}} + \frac{1-\alpha}{2}(\|Xx\|^{2r} + \|Yx\|^{2r})$$

$$(\text{by the arithmetic} - \text{geometric mean inequality})$$

$$= \frac{\alpha}{2}\left(\langle |X|^2x, x\rangle^{\frac{r}{2}} + \langle |Y|^2x, x\rangle^{\frac{r}{2}}\right)|\langle Y^*Xx, x\rangle|^{\frac{r}{2}} + \frac{1-\alpha}{2}(\langle |X|^2x, x\rangle^r + \langle |Y|^2x, x\rangle^r)$$

$$\le \frac{\alpha}{2}(\langle |X|^rx, x\rangle + \langle |Y|^rx, x\rangle)|\langle Y^*Xx, x\rangle|^{\frac{r}{2}} + \frac{1-\alpha}{2}(\langle |X|^rx, x\rangle^2 + \langle |Y|^rx, x\rangle^2)(\text{by Lemma 2.1})$$

$$= \frac{\alpha}{2}\langle (|X|^r + |Y|^r)x, x\rangle|\langle Y^*Xx, x\rangle|^{\frac{r}{2}} + \frac{1-\alpha}{2}|\langle (|X|^r + i|Y|^r)x, x\rangle|^2.$$

Thus,

$$w^r(Y^*X) = \sup_{||x||=1} |\langle Y^*Xx, x\rangle|^r$$

$$\leq \frac{\alpha}{2} \| |X|^r + |Y|^r \| w^{\frac{r}{2}}(Y^*X) + \frac{1-\alpha}{2} w^2(|X|^r + i|Y|^r).$$

Remark 2.7 *The upper bound presented in the above theorem is smaller than the upper bound given in the inequality (1.6). To see this, note that for every $\alpha \in [0,1]$ and $r \geq 2$, we have*

$$w^r(Y^*X) \leq \frac{\alpha}{2} \| |X|^r + |Y|^r \| w^{\frac{r}{2}}(Y^*X) + \frac{1-\alpha}{2} w^2(|X|^r + i|Y|^r) \quad \text{(by Theorem2.6)}$$

$$\leq \frac{\alpha}{2} \| |X|^r + |Y|^r \| w^{\frac{r}{2}}(Y^*X) + \frac{1-\alpha}{2} \| |X|^{2r} + |Y|^{2r} \| \text{(by Lemma 2.6)}$$

$$\leq \frac{\alpha}{4} \| (|X|^r + |Y|^r)^2 \| + \frac{1-\alpha}{2} \| |X|^{2r} + |Y|^{2r} \| \quad \text{(by the inequality (1.6))}$$

$$\leq \frac{\alpha}{2} \| |X|^{2r} + |Y|^{2r} \| + \frac{1-\alpha}{2} \| |X|^{2r} + |Y|^{2r} \| \quad \text{(by Lemma 2.5)}$$

$$= \frac{1}{2} \| |X|^{2r} + |Y|^{2r} \|.$$

The next result in this paper refine [Aujla & Silva, 2003, Theorem 2.9].

Theorem 2.8 *Let $X \in \mathcal{B}(\mathcal{H})$. Then for every $\alpha \in [0,1]$ and $r \geq 2$, we have*

$$w^r(X) \leq \frac{\alpha}{2} w^2\left(|X|^{\frac{r}{2}} + i|X^*|^{\frac{r}{2}}\right) + \frac{1-\alpha}{2} w^{\frac{r}{2}}(X) \left\| |X|^{\frac{r}{2}} + |X^*|^{\frac{r}{2}} \right\|$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. Then we have

$$|\langle Xx, x\rangle|^r = \alpha |\langle Xx, x\rangle|^r + (1-\alpha)|\langle Xx, x\rangle|^r$$

$$\leq \alpha \langle |X|x, x\rangle^{\frac{r}{2}} \langle |X^*|x, x\rangle^{\frac{r}{2}} + (1-\alpha)|\langle Xx, x\rangle|^{\frac{r}{2}} \langle |X|x, x\rangle^{\frac{r}{4}} \langle |X^*|x, x\rangle^{\frac{r}{4}}$$

$$\text{(by the Mixed Schwarz inequality)}$$

$$\leq \frac{\alpha}{2} \left(\langle |X|x, x\rangle^r + \langle |X^*|x, x\rangle^r\right) + \frac{1-\alpha}{2} |\langle Xx, x\rangle|^{\frac{r}{2}} \left(\langle |X|x, x\rangle^{\frac{r}{2}} + \langle |X^*|x, x\rangle^{\frac{r}{2}}\right)$$

$$\text{(by the arithmetic} - \text{geometric mean inequality)}$$

$$\leq \frac{\alpha}{2} \left(\left\langle |X|^{\frac{r}{2}}x, x \right\rangle^2 + \left\langle |X^*|^{\frac{r}{2}}x, x \right\rangle^2 \right) + \frac{1-\alpha}{2} |\langle Xx, x\rangle|^{\frac{r}{2}} \left\langle \left(|X|^{\frac{r}{2}} + |X^*|^{\frac{r}{2}}\right)x, x \right\rangle \quad \text{(by Lemma2.1)}$$

$$= \frac{\alpha}{2} \left| \left\langle \left(|X|^{\frac{r}{2}} + i|X^*|^{\frac{r}{2}}\right)x, x \right\rangle \right|^2 + \frac{1-\alpha}{2} |\langle Xx, x\rangle|^{\frac{r}{2}} \left\langle \left(|X|^{\frac{r}{2}} + |X^*|^{\frac{r}{2}}\right)x, x \right\rangle.$$

Thus,

$$w^r(X) = \sup_{||x||=1} |\langle Xx, x\rangle|^r$$

$$\leq \frac{\alpha}{2} w^2\left(|X|^{\frac{r}{2}} + i|X^*|^{\frac{r}{2}}\right) + \frac{1-\alpha}{2} w^{\frac{r}{2}}(X) \left\| |X|^{\frac{r}{2}} + |X^*|^{\frac{r}{2}} \right\|.$$

Remark 2.9 *The upper bound presented in the above theorem is smaller than the upper bound given in the inequality (1.3). To see this, note that for every* $\alpha \in [0,1]$ *and* $r \geq 2$, *we have*

$$w^r(X) \leq \frac{\alpha}{2} w^2\left(|X|^{\frac{r}{2}} + i|X^*|^{\frac{r}{2}}\right) + \frac{1-\alpha}{2} w^{\frac{r}{2}}(X) \left\| |X|^{\frac{r}{2}} + |X^*|^{\frac{r}{2}} \right\| \text{ (by Theorem 2.9)}$$

$$\leq \frac{\alpha}{2} \| |X|^r + |X^*|^r \| + \frac{1-\alpha}{2} w^{\frac{r}{2}}(X) \left\| |X|^{\frac{r}{2}} + |X^*|^{\frac{r}{2}} \right\| \text{ (by Lemma 2.5)}$$

$$\leq \frac{\alpha}{2} \| |X|^r + |X^*|^r \| + \frac{1-\alpha}{4} \left\| \left(|X|^{\frac{r}{2}} + |X^*|^{\frac{r}{2}}\right)^2 \right\| \text{ (by the inequality (1.3))}$$

$$\leq \frac{\alpha}{2} \| |X|^r + |X^*|^r \| + \frac{1-\alpha}{2} \| |X|^r + |X^*|^r \| \text{ (by Lemma 2.2)}$$

$$= \frac{1}{2} \| |X|^r + |X^*|^r \|.$$

Now, we give an upper bound for the numerical radius of a 2×2 operator matrix which generalize [8, Theorem 2.1].

Theorem 2.10 *Let* $A, B, C, D \in \mathcal{B}(\mathcal{H})$. *Then for every* $r \geq 2$, *we have*

$$w^r\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq 2^{r-1} \max\left\{w^r(A), w^r(D)\right\} + 2^{r-2} \max\left\{w^{\frac{r}{2}}(BC), w^{\frac{r}{2}}(CB)\right\}$$
$$+2^{r-3} \max\left\{\left\| |C|^r + |B^*|^r \right\|, \left\| |B|^r + |C^*|^r \right\|\right\}$$

Proof. Let

$$Y = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \; Y_1 = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \text{ and}$$

$Y_2 = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. Then for every unit vector $x \in \mathcal{H}^{(2)}$, we have

$$|\langle Yx, x\rangle|^r \leq (|\langle Y_1x, x\rangle| + |\langle Y_2x, x\rangle|)^r$$

$$\leq 2^{r-1}|\langle Y_1x, x\rangle|^r + 2^{r-1}|\langle Y_2x, x\rangle|^r \text{ (by the convexity of } t^r, r \geq 2)$$

$$\leq 2^{r-1}|\langle Y_1x, x\rangle|^r + 2^{r-1}|\langle Y_2x, x\rangle\langle x, Y_2^*x\rangle|^{\frac{r}{2}}$$

$$\leq 2^{r-1}|\langle Y_1x, x\rangle|^r + 2^{r-1}\left(\frac{|\langle Y_2^2x, x\rangle|}{2} + \frac{\|Y_2x\|\|Y_2^*x\|}{2}\right)^{\frac{r}{2}} \text{ (by Lemma 2.4)}$$

$$\leq 2^{r-1}|\langle Y_1x, x\rangle|^r + 2^{r-2}\left(|\langle Y_2^2x, x\rangle|^{\frac{r}{2}} + \|Y_2x\|^{\frac{r}{2}}\|Y_2^*x\|^{\frac{r}{2}}\right) \text{ (by the convexity of } t^{\frac{r}{2}}, r \geq 2)$$

$$\leq 2^{r-1}|\langle Y_1x,x\rangle|^r + 2^{r-2}|\langle Y_2^2x,x\rangle|^{\frac{r}{2}} + 2^{r-3}(\|Y_2x\|^r + \|Y_2^*x\|^r)$$

$$\text{(by the arithmetic} - \text{geometric mean inequality)}$$

$$\leq 2^{r-1}|\langle Y_1x,x\rangle|^r + 2^{r-2}|\langle Y_2^2x,x\rangle|^{\frac{r}{2}} + 2^{r-3}\langle(|Y_2|^r + |Y_2^*|^r)x,x\rangle.$$

Therefore,

$$w^r(Y) = \sup_{\|x\|=1} |\langle Yx,x\rangle|^r$$

$$\leq 2^{r-1}w^r(Y_1) + 2^{r-2}w^{\frac{r}{2}}(Y_2^2) + 2^{r-3}\||Y_2|^r + |Y_2^*|^r\|$$

$$= 2^{r-1}\max\{w^r(A),w^r(D)\} + 2^{r-2}\max\left\{w^{\frac{r}{2}}(BC),w^{\frac{r}{2}}(CB)\right\}$$

$$+2^{r-3}\max\{\||C|^r + |B^*|^r\|,\||B|^r + |C^*|^r\|\}.$$

There are many upper bounds for the numerical radii of Hilbert space operators that can be obtained from Theorem 2.12. The following results demonstrate some of these upper bounds.

Corollary 2.11 *Let* $A,B \in \mathcal{B}(\mathcal{H})$., *Then for every* $r \geq 2$, *we have*

$$\max\left\{w^r(A-B),w^r(A+B)\right\} = w^r\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right)$$

$$\leq 2^{r-1}w^r(A) + 2^{r-2}w^{\frac{r}{2}}(B^2) + 2^{r-3}\left\||B|^r + |B^*|^r\right\|.$$

By setting $A = 0$ and $B = X$ in the above corollary we have the following result.

Corollary 2.12 *Let* $X \in \mathcal{B}(\mathcal{H})$. *Then for every* $r \geq 2$, *we have*

$$w^r(X) = w^r\left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}\right)$$

$$\leq \frac{1}{2}w^{\frac{r}{2}}(X^2) + \frac{1}{4}\left\||X|^r + |X^*|^r\right\|$$

$$\leq \frac{1}{2}\left\||X|^r + |X^*|^r\right\|.$$

To prove Theorem 2.16, we need the following lemma which can be found in (Al-Dolat & Al-Zoubi, 2023).

Lemma 2.13 *Let* $u,v,w \in \mathcal{H}$ *with* $\|w\| = 1$. *Then*

$$|\langle u,w\rangle\langle w,v\rangle|^r \leq \frac{1}{2}\|u\|^r\|v\|^r + \frac{\alpha}{2}\|u\|^{\frac{r}{2}}\|v\|^{\frac{r}{2}}|\langle u,v\rangle|^{\frac{r}{2}} + \frac{1-\alpha}{2}|\langle u,v\rangle|^r$$

For every $r \geq 1$ and $\alpha \in [0,1]$.

Now, we present new upper bound for the numerical radius of the off-diagonal of a 2×2 operator matrix.

Theorem 2.14 *Let $B, C \in \mathcal{B}(\mathcal{H})$. Then*

$$w^{2r}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \le \frac{1}{4}\max\left\{w^2(|C|+i|B^*|), w^2(|B|+i|C^*|)\right\}$$

$$+\frac{\alpha}{4}\max\left\{\left\||C|^r+|B^*|^r\right\|, \left\||B|^r+|C^*|^r\right\|\right\}\max\left\{w^{\frac{r}{2}}(BC), w^{\frac{r}{2}}(CB)\right\}$$

$$+\frac{1-\alpha}{2}\max\left\{w^r(BC), w^r(CB)\right\},$$

For every $r \ge 2$ and $\alpha \in [0,1]$.

Proof. Let

$$Y = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}.$$

Then for every $x \in \mathcal{H}^{(2)}$ with $||x|| = 1$, we have

$$|\langle Yx, x\rangle|^{2r} = |\langle Yx, x\rangle\langle x, Y^*x\rangle|^r$$

$$\le \tfrac{1}{2}\|Yx\|^r\|Y^*x\|^r + \tfrac{\alpha}{2}\|Yx\|^{\frac{r}{2}}\|Y^*x\|^{\frac{r}{2}}|\langle Yx, Y^*x\rangle|^{\frac{r}{2}} + \tfrac{1-\alpha}{2}|\langle Yx, Y^*x\rangle|^r \quad \text{(by Lemma 2.15)}$$

$$\le \tfrac{1}{4}(\|Yx\|^{2r} + \|Y^*x\|^{2r}) + \tfrac{\alpha}{4}(\|Yx\|^r + \|Y^*x\|^r)|\langle Y^2x, x\rangle|^{\frac{r}{2}} + \tfrac{1-\alpha}{2}|\langle Y^2x, x\rangle|^r$$

$$\text{(by the arithmetic} - \text{geometric mean inequality)}$$

$$\le \tfrac{1}{4}(\langle |Y|^r x, x\rangle^2 + \langle |Y^*|^r x, x\rangle^2) + \tfrac{\alpha}{4}\langle(|Y|^r + |Y^*|^r)x, x\rangle|\langle Y^2x, x\rangle|^{\frac{r}{2}} + \tfrac{1-\alpha}{2}|\langle Y^2x, x\rangle|^r$$

$$\text{(by Lemma 2.1)}$$

$$= \tfrac{1}{4}|\langle(|Y|^r + i|Y^*|^r)x, x\rangle|^2 + \tfrac{\alpha}{4}\langle(|Y|^r + |Y^*|^r)x, x\rangle|\langle Y^2x, x\rangle|^{\frac{r}{2}} + \tfrac{1-\alpha}{2}|\langle Y^2x, x\rangle|^r.$$

Thus,

$$w^{2r}(Y) = \sup_{||x|=1} |\langle Yx, x\rangle|^{2r}$$

$$\le \frac{1}{4}\max\left\{w^2(|C|+i|B^*|), w^2(|B|+i|C^*|)\right\}$$

$$+\tfrac{\alpha}{4}\max\{\||C|^r + |B^*|^r\|, \||B|^r + |C^*|^r\|\}\max\left\{w^{\frac{r}{2}}(BC), w^{\frac{r}{2}}(CB)\right\}$$

$$+\frac{1-\alpha}{2}\max\{w^r(BC), w^r(CB)\}.$$

As special case of Theorem 2.16, we have the following refinement of the inequality (1.4).

Corollary 2.15 *Let* $X \in \mathcal{B}(\mathcal{H})$. *Then for every* $\alpha \in [0,1]$ *and* $r \geq 2$ *we have*

$$w^{2r}(X) \leq \frac{1}{4}w^2(|X|^r + i|X^*|^r) + \frac{\alpha}{4}\||X|^r + |X^*|^r\|w^{\frac{r}{2}}(X^2) + \frac{1-\alpha}{2}w^r(X^2)$$

$$\leq \frac{1+\alpha}{4}\||X|^{2r} + |X^*|^{2r}\| + \frac{1-\alpha}{2}w^r(X^2).$$

Proof. We have

$$w^{2r}(X) = w^{2r}\left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}\right) \text{ (by Lemma 2.3)}$$

$$\leq \frac{1}{4}w^2(|X|^r + i|X^*|^r) + \frac{\alpha}{4}\||X|^r + |X^*|^r\|w^{\frac{r}{2}}(X^2) + \frac{1-\alpha}{2}w^r(X^2) \quad \text{(by Theorem 2.16)}$$

$$\leq \frac{1}{4}\||X|^{2r} + |X^*|^{2r}\| + \frac{\alpha}{4}\||X|^r + |X^*|^r\|w^{\frac{r}{2}}(X^2) + \frac{1-\alpha}{2}w^r(X^2) \quad \text{(by Lemma 2.5)}$$

$$\leq \frac{1}{4}\||X|^{2r} + |X^*|^{2r}\| + \frac{\alpha}{8}\|(|X|^r + |X^*|^r)^2\| + \frac{1-\alpha}{2}w^r(X^2) \quad \text{(by the inequality (1.6))}$$

$$\leq \frac{1+\alpha}{4}\||X|^{2r} + |X^*|^{2r}\| + \frac{1-\alpha}{2}w^r(X^2) \quad \text{(by Lemma 2.2).}$$

In the following result, we find a new an upper bound for the numerical radius of a 2×2 operator matrix.

Corollary 2.16 *Let* $A, B, C, D \in \mathcal{B}(\mathcal{H})$. *Then for every* $r \geq 2$ *and* $\alpha \in [0,1]$, *we have*

$$w^{2r}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq 2^{2r-1}\max\left\{w^{2r}(A), w^{2r}(D)\right\}$$

$$+2^{2r-3}\max\left\{w^2(|C| + i|B^*|), w^2(|B| + i|C^*|)\right\}$$

$$+\alpha 2^{2r-3}\max\{\||C|^r + |B^*|^r\|, \||B|^r + |C^*|^r\|\}\max\left\{w^{\frac{r}{2}}(BC), w^{\frac{r}{2}}(CB)\right\}$$

$$+(1-\alpha)2^{2r-2}\max\{w^r(BC), w^r(CB)\}.$$

Proof. By the convexity the of t^{2r} and Theorem 2.16 we get

$$w^{2r}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \left(w\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right)^{2r}$$

$$\leq 2^{2r-1}w^{2r}\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + 2^{2r-1}w^{2r}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)$$

$$\leq 2^{2r-1}\max\{w^{2r}(A), w^{2r}(D)\} + 2^{2r-3}\max\{w^2(|C|^r + i|B^*|^r), w^2(|B|^r + i|C^*|^r)\}$$

$$+\alpha 2^{2r-3}\max\{\||C|^r + |B^*|^r\|, \||B|^r + |C^*|^r\|\}\max\left\{w^{\frac{r}{2}}(BC), w^{\frac{r}{2}}(CB)\right\}$$

$$+(1-\alpha)2^{2r-2}\max\{w^r(BC), w^r(CB)\}.$$

The following result presents an upper bound for the numerical radius of the sum of operators.

Corollary 2.17 *Let* $A, B \in \mathcal{B}(\mathcal{H})$. *Then for every* $r \geq 2$ *and* $\alpha \in [0,1]$ *we have*

$$\max\left\{w^{2r}(A-B), w^{2r}(A+B)\right\} = w^{2r}\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right)$$

$$\leq 2^{2r-1}w^{2r}(A) + 2^{2r-3}w^2(|B|^r + i|B^*|^r) + \alpha 2^{2r-3}\||B|^r + |B^*|^r\| w^{\frac{r}{2}}(B^2)$$

$$+(1-\alpha)2^{2r-2}w^r(B^2).$$

To prove Theorem 2.22, we need the following lemma which can be found in (Al-Dolat & Al-Zoubi, 2023).

Lemma 2.18 *Let* $u, v, w \in \mathcal{H}$ *with* $||w|| = 1$. *Then*

$$|\langle u, w\rangle\langle w, v\rangle|^r \leq \frac{1}{4}\|u\|^r\|v\|^r + \frac{2+\alpha}{4}\|u\|^{\frac{r}{2}}\|v\|^{\frac{r}{2}}|\langle u, v\rangle|^{\frac{r}{2}} + \frac{1-\alpha}{4}|\langle u, v\rangle|^r$$

where $\alpha \in [0,1]$ and $r \geq 2$.

Now, we can state the following result in this paper as follows.

Theorem 2.19 *Let* $B, C \in \mathcal{B}(\mathcal{H})$. *Then for every* $\alpha \in [0,1]$ *and* $r \geq 2$ *we have*

$$w^{2r}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{8}\max\left\{w^2(|C|^r + i|B^*|^r), w^2(|B|^r + i|C^*|^r)\right\}$$

$$+\frac{2+\alpha}{8}\max\{\||C|^r + |B^*|^r\|, \||B|^r + |C^*|^r\|\}\max\left\{w^{\frac{r}{2}}(BC), w^{\frac{r}{2}}(CB)\right\}$$

$$+\frac{1-\alpha}{4}\max\{w^r(BC), w^r(CB)\}.$$

Proof. Let

$$Y = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}.$$

Then for every $x \in \mathcal{H}^{(2)}$ with $||x|| = 1$ we have

$$|\langle Yx, x\rangle|^{2r} = |\langle Yx, x\rangle\langle x, Y^*x\rangle|^r$$

$$\leq \frac{1}{4}\|Yx\|^r\|Y^*x\|^r + \frac{2+\alpha}{4}\|Yx\|^{\frac{r}{2}}\|Y^*x\|^{\frac{r}{2}}|\langle Yx, Y^*x\rangle|^{\frac{r}{2}} + \frac{1-\alpha}{4}|\langle Yx, Y^*x\rangle|^r \quad \text{(by Lemma 2.21)}$$

$$\leq \frac{1}{8}(\|Yx\|^{2r} + \|Y^*x\|^{2r}) + \frac{2+\alpha}{8}(\|Yx\|^r + \|Y^*x\|^r)|\langle Y^2x, x\rangle|^{\frac{r}{2}} + \frac{1-\alpha}{4}|\langle Y^2x, x\rangle|^r$$

$$\text{(by the arithmetic} - \text{geometric mean inequality)}$$

$$\leq \frac{1}{8}|\langle(|Y|^r + i|Y^*|^r)x, x\rangle|^2 + \frac{2+\alpha}{8}\langle(|Y|^r + |Y^*|^r)x, x\rangle|\langle Y^2x, x\rangle|^{\frac{r}{2}} + \frac{1-\alpha}{4}|\langle Y^2x, x\rangle|^r$$

$$\text{(by Lemma 2.1).}$$

Therefore,

$$w^{2r}(Y) = \sup_{||x||=1}|\langle Yx, x\rangle|^{2r}$$

$$\leq \frac{1}{8}|\langle(|Y|^r + i|Y^*|^r)x, x\rangle|^2 + \frac{2+\alpha}{8}\||Y|^r + |Y^*|^r\|w^{\frac{r}{2}}(Y^2) + \frac{1-\alpha}{4}w^r(Y^2)$$

$$= \frac{1}{8}\max\{w^2(|C|^r + i|B^*|^r), w^2(|B|^r + i|C^*|^r)\}$$

$$+ \frac{2+\alpha}{8}\max\{\||C|^r + |B^*|^r\|, \||B|^r + |C^*|^r\|\}\max\left\{w^{\frac{r}{2}}(BC), w^{\frac{r}{2}}(CB)\right\}$$

$$+ \frac{1-\alpha}{4}\max\{w^r(BC), w^r(CB)\}.$$

Corollary 2.20 *Let* $X \in \mathcal{B}(\mathcal{H})$. *Then for every* $\alpha \in [0,1]$ *and* $r \geq 2$, *we have*

$$w^{2r}(X) \leq \frac{1}{8}w^2(|X|^r + i|X^*|^r) + \frac{2+\alpha}{8}\||X|^r + |X^*|^r\|w^{\frac{r}{2}}(X^2) + \frac{1-\alpha}{4}w^r(X^2)$$

$$\leq \frac{1}{2}\||X|^{2r} + |X^*|^{2r}\|.$$

Proof. We have

$$w^{2r}(X) = w^{2r}\left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}\right)$$

$$\text{(by Lemma 2.3)}$$

$$\leq \frac{1}{8}w^2(|X|^r + i|X^*|^r) + \frac{2+\alpha}{8}\||X|^r + |X^*|^r\|w^{\frac{r}{2}}(X^2) + \frac{1-\alpha}{4}w^r(X^2) \quad \text{(by Theorem 2.22)}$$

$$\leq \frac{1}{8}\||X|^{2r} + |X^*|^{2r}\| + \frac{2+\alpha}{8}\||X|^r + |X^*|^r\|w^{\frac{r}{2}}(X^2) + \frac{1-\alpha}{4}w^r(X^2) \quad \text{(by Lemma 2.5)}$$

$$\leq \frac{1}{8}\||X|^{2r} + |X^*|^{2r}\| + \frac{2+\alpha}{16}\|(|X|^r + |X^*|^r)^2\| + \frac{1-\alpha}{8}\||X|^{2r} + |X^*|^{2r}\| \quad \text{(by the inequality (1.6))}$$

$$\leq \frac{1}{8}\||X|^{2r} + |X^*|^{2r}\| + \frac{2+\alpha}{8}\||X|^{2r} + |X^*|^{2r}\| + \frac{1-\alpha}{8}\||X|^{2r} + |X^*|^{2r}\| \quad \text{(by Lemma 2.2)}$$

$$= \frac{1}{2}\||X|^{2r} + |X^*|^{2r}\|.$$

Scientific Ethics Declaration

The author declares that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

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References

Abu-Omar, A. & Kittaneh, F. (2015). Upper and lower bounds for the numerical radius with an application to involution operators, *Rocky Mountain Journal Math*., *45*(4), 1055-1065.

Al-Dolat, M., & Al-Zoubi, K. (2023). Improved and refined numerical radius inequalities for Hilbert space operators.*https://assets.researchsquare.com/files/rs2668438/v1/a5b67b067e6f6fddcf09cb22.pdf?c=1678771282*

Al-Dolat, M., Al-Zoubi, K., Ali, M., & Bani-Ahamed, F. (2016). General numerical radius inequalities for matrices of operators, *Open Math., 4*, 1-9.

Al-Dolat, M., & Jaradat, I. (2023). A refinement of the Cauchy-Shwarz inequality accompanied by new numerical radius upper bounds. *Filomat*, *37*, 971-977.

Al-Dolat, M., & Kittaneh, F. (2023). Upper bounds for the numerical radii of powers of Hilbert space operators. *Quaestiones Mathematicae,* 1-12.

Aujla, J., & Silva, F. (2003). Weak majorization inequalities and convex functions. *Linear Algebra Appl., 369,* 217-233.

Bani-Domi W., & Kittaneh, F. (2021). Refined and generalized numerical radius inequalities for 2x2 operator matrices. *Linear Algebra Appl*. 364-380.

Bani-Domi, W. & Kittaneh F. (2021). Norm and numerical radius inequalities for Hilbert space operators. *Linear Multilinear Algebra, 69,* 934-945.

Buzano, M.L. (1974). Generalizzazione della diseguaglianza di Cauchy-Schwarz, (Italian). *Rend Sem Mat Univ e Politech Torino, 31,* 405-409.

Dragomir, S.S. (2009). Power inequalities for the numerical radius of a product of two operators in Hilbert spaces. *Sarajevo J Math., 18,* 269-278.

El-Haddad, M., & Kittaneh, F. (2007). Numerical radius inequalities for Hilbert space operators. *II Studia Math., 182,* 133-140.

Kittaneh, F. & Moradi, H. R. (2020). Cauchy-Schwarz type inequalities and applications to numerical radius inequalities. *Mathematical Inequalities and Applications, 23,* 1117-1125.

Kittaneh, F. (1988). Notes on some inequalities for Hilbert space operators. *Pub1. Reserach Inst. Math. Science, 24*, 283-293.

Kittaneh, F. (2003). A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix. *Studia Math., 158,* 11-17.

Kittaneh, F. (2005) Numerical radius inequalities for Hilbert space operators. *Studia Math., 168,* 73-80.

Moradi, H. ,& Sababheh, M. (2021). New estimates for the numerical radius. *Filomat, 35,* 4957-4962.

Halmos, P.R. (1982). *A Hilbert space Problem Book*, (2nd ed).New York, NY:Springer.

Author Information

Mohammed Al-Dolat
Jordan University of Science and Technology
Irbid, Jordan
Contact e-mail: mmaldolat@just.edu.jo

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