The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM)
ISSN: 2602-3199

The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM), 2023

Volume 22, Pages 171-175

ICBASET 2023: International Conference on Basic Sciences, Engineering and Technology

Useful Ideas on the Numerical Techniques Used for the Solution of the Two-Point Boundary Value Problems of Ordinary Differential Equations

Mohammad H. AL-TOWAIQ
Jordan University of Science and Technology

Abstract: In this paper, we give a preview of some numerical techniques used for the solution of the two-point boundary value problems of ordinary differential equations. We discuss the difficulties one faces in using them, which need further investigation, and we suggest some ideas to overcome these difficulties.

Keywords: Ordinary differential equations, Two-point boundary Value problem.

Introduction

Two-point boundary value problems (BVP) of ordinary differential equations describe many physical phenomena in science and engineering such as electromagnetism, astronomy, mechanical vibration analysis, and many other topics. For our discussion, we will consider the following two-point BVP for ordinary differential equations

$$x''(t) = f\big(t, x(t), x'(t)\big), \tag{1}$$

with boundary conditions BCs) at two distinct points *a* and *b* of the form

$$x(a) = \propto, x(b) = \beta \tag{2}$$

This type of equation often poses a difficult challenge to the numerical analyst, since most of them are nonlinear. In most cases, analytical solutions are not available or difficult to find. Therefore, numerical methods can be used to find an approximate solution. There are different numerical methods and their modifications used to compute an approximate solution of the two-point BVP such as the shooting method (SM), finite difference methods (FDM), finite element method (FEM), collection method (CM), Galerkin method (GM), and Least Squares method (LSM), (Linz & Wang, 2003). Recently, many researchers presented certain techniques based on the above methods to compute accurate solutions to the two-point BVP.

The Shooting Method

The idea of the shooting method is to convert the BVP to an initial value problem (IVP), (Adam, & Hashim, 2014). Then we can employ a suitable algorithm for the solution of the IVP. The first initial value is $x(a) = \propto$, then we assign a real value ω to the missing initial condition $x'(a) = \omega$. Hence, the value of x at b is a function of ω, say φ(ω). So, the procedure goes as follows:

1. Compute the function φ(ω) using the IVP (1) with the initial conditions $x(a)=\propto$ and $x'(a)=\omega$.
2. Let $\varphi(\omega) = x(b)$.
3. Modify ω iteratively until we find a value which satisfies $\varphi(\omega)\cong\beta$

Difficulties:

1. The iterative approach in step 3, especially in the nonlinear case, converts to a root-finding in a single unknown ω, but not all of the root-finding methods are easy to apply. In particular, it is hard to use Newton's method since we do not have an explicit expression for the given function. Using the secant method will work fine, but the computational time is very expensive. Instead, may we can use the modified Quasi-Newton's method or the modified Broyden's method (Al-Towaiq & Abu-Hour, 2016, 2017).
2. The analysis of the shooting method is complicated by the fact that we do not know how the errors in the solution of the IVP affect the accuracy of the solution of the BVP.

Some variants have been proposed to extend the shooting method, see (Filipov et al., 2017; Perfilieva et al., 2017; Edun & Akinlabi, 2021; Arefin et al., 2022).

Finite Differences Method

The FDM is one of the most popular techniques for the solution of many two-point BVP. The method replaces the derivatives with finite differences to produce a finite system of equations. In the linear case, we obtain a linear system

$$Ax = b \tag{3}$$

This system is easy to solve even for very small mesh sizes. But, there are some limitations:
1. There will be a serious difficulty with nonlinear discretization.
2. The error analysis is generally hard.
3. The FDM is most of the time consistent, but the resolution of the stability is not obvious and requires good technical skills, utilizing the nature of matrix A.

However, these difficulties do not prevent researchers from using the FDM, i.e. (Cicelia, 2014; Ahmad & Charan, 2019).

Collocation Method

The idea of this method is to choose a finite-dimensional space of candidate solutions and a number of points in the domain (called *collocation points*) and to select that solution that satisfies the given equation at the collocation points, as follows

1. Write an approximate solution as a linear combination of a set of basis functions

$$x_n(t) = \sum_{k=0}^{n} c_k \varphi_k(t) \tag{4}$$

2. c_k and $\varphi_k(t)$ must be selected so that they can give an approximate solution to the BVP accurately, if possible.

Normally, equation (4) does not give equality, no matter how we select the $c_k's$ and $\varphi_k(t)'s$. The best way to do this is to select a finite number of collection points for which the equation is satisfied at these points. Several researchers attempted to use polynomial approximation, but this leads to poor conditioning. Instead, they prefer orthogonal polynomials such as the Chebyshev polynomials (Ehrenstein & Peyret, 1989; Soliman et al., 2014), while others used radial basis functions (Hu et al., 2007;Al-Towaiq et al., 2018). Nevertheless, the approach has limitations, i.e. convergence analysis and selecting the collocation points. This is a disadvantage, especially with a nonlocal basis, because the resulting matrix will be full. Also, to overcome these issues, some researchers prefer to take a local basis such as the cubic B-splines (Khalifa at al., 2008; Mittal & Jain, 2012).

Least Squares Method (LSM)

The LSM has been developed with (4) by considering the residual

$$R(x_n(t),t) = x_n''(t) - f(t, x(t), \quad .ş$$

The coefficients in the CM are chosen to satisfy the BCs and $R(x_n,\ t_i) = 0, i = 1,2,\dots,n-1$, where the t_i's are the collocation points. So, the $c_k{}'s$ in (4) is chosen to minimize the integral of the square of the residual,

$$\|R\|_2^2 = \int_a^b R^2(x_n(t),t)dt \tag{6}$$

Substitute (4) into (6), then we can obtain the $c_k{}'s$ by solving the following linear system

$$\mathrm{A}c = b, \tag{7}$$

where

$$a_{i,j} = \int_a^b h_{i-1}(t)\, h_{j-1}(t)dt,\ \ h_i(t) = \ \varphi_i''(t) - f(t, \varphi_i(t), \varphi_i'(t)),$$
$$b_i = \int_a^b g(t)h_{i-1}(t)dt, \tag{8}$$

where $g(t)$ is the nonhomogeneous term in the DE.

The problem here, (6) does not account for the BC. To overcome this problem, we must transform the differential equation to have homogeneous BCs. Then choose the basis functions $\varphi_k(t)$, so (4) will satisfy the homogeneous BCs, which implies that (7) will give the LS solution. For example, for polynomial approximations, we may use Chebysheve polynomials, for B-splines, may the interior splines functions satisfy the homogeneous BCs, but the end ones do not. For these, may we can use a combination of several B-splines. Unfortunately, this will cause a complication in the integrals, (8), which compute the coefficients matrix A and the right-hand side of (7). Thus, the LSM becomes very expansive. An alternative of this is to use collocation, (Humboldt, 1988), which in most cases (7) becomes an overdetermined system and we cannot expect to have a solution. Hence, we can use the LS sense, which leads to another method that falls between LSM and CM called the least square collocation method (LSCM), (Kee et al., 2007; Shapeev et al., 2018; Paffuti, 2019) . The LSCM tends to be easier and more efficient than the LSM. But, this needs special treatment from the BCs. However, it turns out that whatever techniques are being used, the BC's satisfied approximately which leads to an inaccurate solution. To avoid this, may we can weigh the BCs as the collocation points increase?

Galerkin Method (GM)

This method chooses the coefficients $c_k{}'s$ of equation (4) so that the residual is orthogonal to $\varphi_k(t)$, k =0, 1, …,n, that is

$$\int_a^b R(x_n(t),t)\varphi_k(t)dt = 0,\ \ k = 0,1,\dots,n. \tag{9}$$

This leads to a system similar to (7) with

$$a_{i,j} = \int_a^b h_{i-1}(t)\, \varphi_{j-1}(t)dt,\ \ h_i(t) = \ \varphi_i''(t) - f(t, \varphi_i(t), \varphi_i'(t)),$$
$$b_i = \int_a^b g(t)\varphi_{i-1}(t)dt. \tag{10}$$

For this to work, the BCs of the problem should be homogeneous to be satisfied by all the functions in (4). Extensive work has been done on the analysis of the method, (Pan et al., 2005; Cicelia, 2014; Zavalani, 2015; Anulo. et al., 2017; Paffuti, 2019;Wang & Zhao, 2019), but it still is an attractive and challenging area of research. GM has a powerful numerical tool for finding fast and accurate solutions. However, when more terms are used in the trial solution, the GM presents more analytical difficulties than the other methods.

Conclusion and Future Work

The SM is very efficient, applicable, and easy to use for engineering and applied sciences problems.

The LSM and GM are very competitive with the FDM only under certain conditions. A few practical problems remain to be solved, for example

1. Improvement of the SM to solve two-point BVP for fourth-order equations, such as the Euler-Bernoulli beam equation $(a(x)u(x)'')'' + q(x)u(x) = f(x)$.
2. The FDM needs efficient improvements to handle the general boundary conditions by carrying out numerical experiments to form conjectures on the stability and order of convergence of the improvement technique.
3. Making the FDM more automatic by letting the algorithm choose the discretization for the problem in hand. In addition, is the selection of the discretization locally reduces the global error efficiently?
4. In the Collocation-Least Squares technique, carry out experimentation to see if the increasing number of collocation points will affect the estimation of the BC's and the computational complexity.
5. In general, finding the connection between the local and the global error is complicated and needs more investigation.
6. Implementing the above methods to find the solution of the problem under the general homogeneous BC's:
   $c_0x(a) + c_1x'(a) = 0$
   $d_0x(b) + d_1x'(a) = 0$

Scientific Ethics Declaration

The author declares that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the author.

Acknowledgments or Notes

* This article was presented as an oral presentation at the International Conference on Basic Sciences, Engineering, and Technology (www.icbaset.net) held in Marmaris /Turkey on April 27-30, 2023.

References

Adam, B., & Hashim, M. H. (2014). Shooting method in solving boundary value problem, *International Journal of Research and Reviews in Applied Sciences 21.1*, 8-30.

Ahmad, N., & Charan, S. (2019). A comparative study of numerical solutions of second order ordinary differential equations with boundary value problems by shooting method & finite difference method. *Numerical Solution of Integral Equation, 4* (1), 18-22.

Al-Towaiq, M., & Abu-Hour, Y. (2017). Two improved classes of Broyden' methods for solving nonlinear systems of equations. *Journal of Mathematics and Computer Science*, *17*, 22-31.

Al-Towaiq, M., & Abu-Hour, Y. (2016). Two improved methods based on Broyden's Newton methods for the solution of nonlinear systems of equations. *Journal of Engineering and Applied Sciences*, *11*(11), 2344-2348.

Al-Towaiq, M., Ababnah, A., & Al-Shwayyat, S. (2018). An efficient approach for Solving Fisher's equation using radial basis-collocation technique. *Proceedings of 161st The IIE International Conference, Barcelona*, Spain.

Anulo, A., Kibret, A., Gonfa, G., & Negassa,G. (2017). Numerical solution of linear second order differential equations with mixed boundary conditions by Galerkin method. *Mathematics and Computer Science*, *2*(5), 66-78.

Arefin, M. A., Nishu,M.A., Dhali, M.N., & Uddin, M.H. (2022). Analysis of reliable solutions to the boundary value Problems by Using ShootMethod. *Mathematical Problems in Engineering*, Article ID 2895023, https://doi.org/10.1155/2022/2895023

Cicelia, J. E. (March 2014). Solution of weighted residual problems by using Galerkin's method, *Indian Journal of Science and Technology*, *7*(3S), 52-54.

Edun, I. F., & Akinlabi, G. O. (2021). Application of the shooting method for the solution of second order boundary value problems. *Journal of Physics: Conference Series,1734.*

Ehrenstein, U., & Peyret, R. (1989). A chebyshev collocation method for the navier–stokes equa-tions with application to double-diffusive convection. *International Journal for Numerical Methods in Fluids, 9*(4), 427–452.

Filipov, S. Gospodinov, I., & Faragó, I. (2017). Shooting-projection method for two-point boundary value problems. *Applied Mathematics Letters*, *72*, 10-15.

Hu, H. Y., Chen, J. S., & HU, W. (2007). A weighted radial basis collocation method for boundary value problems. *International Journal for Numerical Methods in Engineering*, *69*(13), 2736-2757.

Humboldt, H. (1988). On a least-squares collocation method for linear lifferential-algebraic equations. *Numerische Mathematik*, *54*, 79-90.

Kee, B., Liu, G. R., & Lu, C. (2007). A regularized least-squares radial point collocation method (RLS-RPCM) for adaptive analysis. *Computational Mechanic*, *40*(5), 837–853.

Khalifa, A. K., & Alzubaidi, H.M. (2008). A collocation method with cubic B-splines fo solving the MRL equation. *Journal of Computational and Applied Mathematics*, *212*(2), 406-418.

Linz, P., & Wang, R. L. C. (2003). *Exploring numerical methods: An introduction to scientific computing using MATLAB.* Jones & Bartlett Learning.

Mittal, R. C., & Jain R.K. (2012). Numerical solutions of nonlinear Burgers' equation with modified cubicB-splines collocation method. *Applied Mathematics and Computation, 218,*7839-7855.

Paffuti, G. (2019). Galerkin method for discs capacitors*. Mathematics and Computers in Simulation*, *166*, 365-381.

Pan, X. F., Zhang, X., & Lu, M. W. (2005). Meshless Galerkin least- squares method. *Comput Mech 35*, 182-189.

Perfilieva, I., Števuliáková P., & Valášek, R. (2017). Shooting method based on higher degree F-transform. *2017 Joint 17th World Congress of International Fuzzy Systems Association and 9th International Conference on Soft Computing and Intelligent Systems (IFSA-SCIS).*

Shapeev, V., Belyaev,V., Golushko,S., & Idımeshev,S . (2018). New possibilities and applications of the least squares collocation method. In *EPJ Web of Conferences (Vol. 173, p. 01012). EDP Sciences.*

Soliman, M., Al-Zeghayer Y., & Ajbar, A. (2014). A modified orthogonal collocation method for reaction-diffusion problems. *Brazillian Journal of Chemical Engineering,31*(4), 967 - 975.

Wang, S. W., & Xiushao Zhao, X. (2019). An interpolating element-free Galerkin scaled boundary method applied to structural dynamic analysis. *Applied Mathematical Modelling, 75,* 494-505.

Zavalani, G. (2015). A Galerkin finite element method for two-point boundary value problems of ordinary differential equations. *Applied and Computational Mathematics*, *4*(2), 64-68.

Author Information

Mohammad H. Al-Towaiq
Jordan University of Science and Technology
Irbid, Jordan
Contact e-mail: towaiq@just.edu.jo

To cite this article:

Al-Towaiq, M. H. (2023). Useful ideas on the numerical techniques used for the solution of the two-point boundary value problems of ordinary differential equations. *The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM), 22,*171-175.