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On gr-Quasi-Semiprime Submodules

Khaldoun AL-ZOUBI
Jordan University of Science and Technology

Shatha ALGHUEIRI
Jordan University of Science and Technology

Abstract: Let G be a group. A ring R is called a graded ring (or G-graded ring) if there exist additive subgroups R_α of R indexed by the elements $\alpha \in G$ such that $R = \bigoplus_{\alpha\in G} R_\alpha$ and $R_\alpha R_\beta \subseteq R_{\alpha\beta}$ for all $\alpha, \beta \in G$. If an element of R belongs to $h(R) = \cup_{\alpha\in G} R_\alpha$, then it is called a homogeneous. A Left R-module M is said to be *a graded R-module* if there exists a family of additive subgroups $\{M_\alpha\}_{\alpha\in G}$ of M such that $M = \bigoplus_{\alpha\in G} M_\alpha$ and $R_\alpha M_\beta \subseteq M_{\alpha\beta}$ for all $\alpha, \beta \in G$. Also if an element of M belongs to $\cup_{\alpha\in G} M_\alpha = h(M)$, then it is called a homogeneous. A submodule N of M is said to be *a graded submodule of M* if $N = \bigoplus_{\alpha\in G}(N \cap M_\alpha) := \bigoplus_{\alpha\in G} N_\alpha$. Let G be a group with identity e. Let R be a G -graded commutative ring and M a graded R-module. A proper graded submodule S of M is said to be *a graded semiprime (*shortly gr -*semiprime) submodule* if whenever $r^n m \in S$ where $r \in h(R)$, $m \in h(M)$ and $n \in Z^+$, then $rm \in S$. In this work, we introduce the concept of graded quasi-semiprime (shortly gr-quasi-semiprime) submodule as a generalization of gr-semiprime submodule and give some basic properties of these classes of graded submodules. We say that a proper graded submodule S of M is a gr-quasi-semiprime submodule if $(S:_R M) = \{r \in R : rM \subseteq S\}$ is a gr-semiprime ideal of R.

Keywords: Graded quasi-semiprime submodule, Graded semiprime submodule, Graded prime.

Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Graded semiprime submodules of graded modules over graded commutative rings, have been introduced and studied in Farzalipour et al., 2012; Lee et al., 2012; Al-Zoubi et al., 2017). Also, the concept of graded semiprime ideal was introduced by Lee and Varmazyar in Lee (2012) and studied in Farzalipour et al. (2013). Recently, Al-Zoubi et al. (2017) introduced and studied the concept of graded semi-radical of graded submodules in graded modules.

Here, we introduce the concept of graded quasi-semiprime submodules of graded modules over a commutative graded rings as a generalization of graded semiprime submodules and investigate some properties of these classes of graded submodules. Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. Then $(N:_R M)$ is defined as $(N:_R M) = \{r \in R : rM \subseteq N\}$. It is shown in Atani (2006) that if N is a graded submodule of M, then $(N:_R M)$ is a graded ideal of R. The annihilator of M is defined as $(0:_R M)$ and is denoted by $Ann_R(M)$. A proper graded submodule N of M is said to be *a graded semiprime submodule* if whenever $r \in h(R)$, $m \in h(M)$ and $n \in Z^+$ with $r^n m \in N$, then $rm \in N$, (Farzalipour et al., 2012). A proper graded ideal I of R is said to be graded semiprime ideal if whenever $r, s \in h(R)$ and

$n \in Z^+$ with $r^n s \in I$, then $rs \in I$, (Farzalipour et al., 2013). For more information about the properties of graded rings and graded modules see Nastasescu et al. (1982), Nastasescu et al. (2004) and Hazrat (2016).

Results and Discussion

Definition 1. Let R be a G-graded ring and M a graded R-module. A proper graded submodule N of M is said to be *a graded quasi-semiprime submodule of* M if $(N:_R M)$ is a graded semiprime ideal of R.

Theorem 2. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. If N is a graded semiprime submodule of M, then N is a graded quasi-semiprime submodule of M.*

Proof. By Al-Zoubi et al. (2017).□

The next example shows that a graded quasi-semiprime submodule is not necessarily graded semiprime submodule.

Example 3. Let $G = \mathbb{Z}_2$, $R = \mathbb{Z}$ be a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z} \times \mathbb{Z}$ be a graded R-module with $M_0 = \mathbb{Z} \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Z}$. Now, consider a submodule $N = 4\mathbb{Z} \times \{0\}$ of M. Then it is a graded submodule and $(N:_R M) = \{0\}$ is a graded semiprime ideal of R, and so N is a graded quasi-semiprime submodule of R. But the graded submodule N is not graded semiprime submodule of M, since $2^2(3,0) \in N$ but $2(3,0) \notin N$.

Example 4. Let $G = \mathbb{Z}_2$, $R = \mathbb{Z}$ be a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}_8$ be a G-graded R-module with $M_0 = \mathbb{Z}_8$ and $M_1 = \{0\}$. Now, consider a submodule $N =< 4 >$ of M. Then it is a graded submodule and $(N:_R M) = 4\mathbb{Z}$ is not a graded semiprime ideal of R since $2^2 1 = 4 \in 4\mathbb{Z}$ but $2 \cdot 1 = 2 \notin 4\mathbb{Z}$. Then N is not graded quasi-semiprime submodule of M.

Recall that a graded R-module M is called *a graded multiplication* if for each graded submodule N of M, we have $N = IM$ for some graded ideal I of R. If N is graded submodule of a graded multiplication module M, then $N = (N:_R M)M$.

Theorem 5. *Let R be a G-graded ring, M a graded multiplication R-module and N a proper graded submodule of M. Then N is a graded quasi-semiprime submodule of M if and only if N is a graded semiprime submodule of M.*

Proof. By Al-Zoubi et al.(2017)

Theorem 6. *Let R be a G-graded ring, M a graded multiplication R-module and N a proper graded submodule of M. Then the following statements are equivalent:*

1. *N is a graded quasi-semiprime submodule of M.*
2. *If whenever $I^k M \subseteq N$, where I is a graded ideal of R and $k \in \mathbb{Z}^+$, then $IM \subseteq N$.*

Proof. $(i) \Rightarrow (ii)$ By Theorem 5 and Farzalipour et al. (2012).

$(ii) \Rightarrow (i)$ Let $r^k s \in (N:_R M)$ where $r, s \in h(R)$ and $k \in \mathbb{Z}^+$. So $r^k sM \subseteq N$. Let $I = (rs)$ be a graded ideal of R generated by rs. Then $I^k M \subseteq N$. By our assumption we have $IM = (rs)M \subseteq N$. This yields that $rs \in (N:_R M)$. So $(N:_R M)$ is a graded semiprime ideal of R. Therefore N is a graded quasi-semiprime submodule of M. □

Recall that a proper graded ideal I of a G-graded ring R is said to be *a graded prime ideal* if whenever $r, s \in h(R)$ with $rs \in I$, then either $r \in I$ or $s \in I$ Refai et al. (2004). A proper graded ideal J of R is said to be *a graded primary ideal* if whenever $r, s \in h(R)$ with $rs \in J$, then either $r \in J$ or $s^n \in J$ for some $n \in \mathbb{Z}^+$ Refai et al.(2004).

Theorem 7. *Let R be a G-graded ring, M a graded R-module and N a graded quasi-semiprime submodule of M. If $(N:_R M)$ is a graded primary ideal of R, then $(N:_R M)$ is a graded prime ideal of R.*

Proof. Suppose that $(N:_R M)$ is a graded primary ideal of R. Let $rs \in (N:_R M)$ and $r \notin (N:_R M)$. Then $s \in Gr\big((N:_R M)\big)$ as $(N:_R M)$ is a graded primary ideal of R. Hence $s^k \in (N:_R M)$ for some $k \in \mathbb{Z}^+$. Since $(N:_R M)$ is a graded semiprime ideal of R, we have $s \in (N:_R M)$. Therefore $(N:_R M)$ is a graded prime ideal of R. □

Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. The graded envelope submodule $RGE_M(N)$ of N in M is a graded submodule of M generated by the set $GE_M(N) = \{rm : r \in h(R), m \in h(M)$ such that $r^n m \in N$ for some $n \in \mathbb{Z}^+\}$ (Atani et al., 2010).

Theorem 8. *Let R be a G-graded ring, M a graded multiplication R-module and N a proper graded submodule of M. Then N is a graded quasi-semiprime submodule of M if and only if $N = RGE_M(N)$.*

Proof. Suppose that N is a graded quasi-semiprime submodule of M. Then N is a graded semiprime submodule of M by Theorem 5. Clearly, $N \subseteq RGE_M(N)$. Now, let $x \in GE_M(N)$. Then $x = rm$ for some $r \in h(R)$, $m \in h(M)$ and there exists $k \in \mathbb{Z}^+$ such that $r^k m \in N$. Then $rm \in N$ as N is a graded semiprime submodule of M. Hence $GE_M(N) \subseteq N$. This yields that $RGE_M(N) \subseteq N$. Thus $N = RGE_M(N)$. Conversely, suppose that $N = RGE_M(N)$. Let $r \in h(R)$, $m \in h(M)$ and $k \in \mathbb{Z}^+$ such that $r^k m \in N$, so by the definition of the set $GE_M(N)$ we have $rm \in GE_M(N)$. Then $rm \in N$ as $GE_M(N) \subseteq RGE_M(N) = N$, so N is a graded semiprime submodule of M. Therefore N is a graded quasi-semiprime submodule of M by Theorem 2. □

Let R be a G-graded ring and M, M' be two graded R-modules. Let $f: M \to M'$ be an R-module homomorphism. Then f is said to be *a graded homomorphism* if $f(M_\alpha) \subseteq M'_\alpha$ for all $\alpha \in G$ (Nastasescu et al., 2004).

Theorem 9. *Let R be a G-graded ring, M, M' be two graded R-modules and $f: M \to M'$ a graded epimorphism.*

1. *If N is a graded quasi-semiprime submodule of M such that $ker(f) \subseteq N$, then $f(N)$ is a graded quasi-semiprime submodule of M'.*
2. *If N' is a graded quasi-semiprime submodule of M', then $f^{-1}(N')$ is a graded quasi-semiprime submodule of M.*

Proof. (i) Suppose that N is a graded quasi-semiprime submodule of M and $ker(f) \subseteq N$. It is easy to see that $f(N) \neq M'$. Now let $r^k s \in (f(N):_R M')$ where $r, s \in h(R)$ and $k \in \mathbb{Z}^+$, it follows that,

$r^k sM' \subseteq f(N)$. Then $r^k sM' = r^k sf(M) = f(r^k sM) \subseteq f(N)$ since f is an epimorphism. This yields that $r^k sM \subseteq N$ since $ker(f) \subseteq N$, i.e., $r^k s \in (N:_R M)$. Since N is a graded quasi-semiprime submodule of M, we get $rs \in (N:_R M)$, i.e., $rsM \subseteq N$. Hence $f(rsM) = rsf(M) = rsM' \subseteq f(N)$, i.e., $rs \in (f(N):_R M')$. Therefore, $f(N)$ is a graded quasi-semiprime submodule of M'.

(ii) Suppose that N' is a graded quasi-semiprime submodule of M'. It is easy to see that $f^{-1}(N') \neq M$. Let $r^k s \in (f^{-1}(N'):_R M)$ where $r, s \in h(R)$ and $k \in \mathbb{Z}^+$, it follows that, $r^k sM \subseteq f^{-1}(N')$. Then $r^k sf(M) = r^k sM' \subseteq N'$, i.e., $r^k s \in (N':_R M')$. Then $rs \in (N':_R M')$ as N' is a graded quasi-semiprime submodule of M'. So $rsM' = rsf(M) = f(rsM) \subseteq N'$. It follows that $rsM \subseteq f^{-1}(N')$. So $rs \in (f^{-1}(N'):_R M)$. Therefore $f^{-1}(N')$ is a graded quasi-semiprime submodule of M.

Theorem 10. *Let R be a G-graded ring, M a graded R-module and K a proper graded submodule of M. If N is a graded quasi-semiprime submodule of M with $N \subseteq K$ and $(N:_R M)$ is a graded maximal ideal of R, then K is a graded quasi-semiprime submodule of M.*

Proof. Suppose that $N \subseteq K$, it follows that $(N:_R M) \subseteq (K:_R M)$. By (Atani, 2006, Lemma 2.1), $(K:_R M)$ is a proper graded ideal of R. Then $(N:_R M) = (K:_R M)$ as $(N:_R M)$ is a graded maximal ideal of R. This yields that $(K:_R M)$ is a graded semiprime ideal of R. Therefore K is a graded quasi-semiprime submodule of M.

Theorem 11. *Let R be a G-graded ring, M a graded R-module and N and K be two graded quasi-semiprime submodules of M. Then $N \cap K$ is a graded quasi-semiprime submodule of M.*

Proof. Let $r^k s \in (N \cap K:_R M)$ where $r, s \in h(R)$ and $k \in \mathbb{Z}^+$. This yields that $r^k s \in (N:_R M) \cap (K:_R M)$. Since $(N:_R M)$ and $(K:_R M)$ are graded semiprime ideals of R, we have $rs \in (N:_R M) \cap (K:_R M)$ and so $rs \in (N \cap K:_R M)$. Therefore $N \cap K$ is a graded quasi-semiprime submodule of M.

Let R be a G-graded ring and M be a graded R-module, M is called a graded semiprime module if (0) is a graded semiprime submodule of M.

Definition 12. Let R be a G-graded ring and M be a graded R-module. Then M is said to be *a graded quasi-semiprime module* if $Ann_R N$ is a graded semiprime ideal of R, for every non-zero graded submodule N of M.

Theorem 13. *Let R be a G-graded ring and M be a graded R -module. If M is a graded semiprime module, then M is a graded quasi-semiprime module.*

Proof. Suppose that M is a graded semiprime module. Then 0 is a graded semiprime submodule of M. Now, Let N be a non-zero graded submodule of M and $r^k s \in Ann_R N$ where $r, s \in h(R)$ and $k \in \mathbb{Z}^+$. It follows that $r^k sN = 0$. Then $rsN = 0$ as 0 is a graded semiprime submodule of M. Hence $rs \in Ann_R N$, it follows that $Ann_R N$ is a graded semiprime ideal of R. Therefore M is a graded quasi-semiprime module.

Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

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Author Information

Khaldoun Al-Zoubi	**Shatha Alghueiri**
Jordan University of Science and Technology	Jordan University of Science and Technology
Irbid, Jordan	Irbid, Jordan
Contact e-mail: kfzoubi@just.edu.jo	

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