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Determination of the Coefficient of Thermal Expansion in a Non-Linear Elastic Rod of Two Concentric Layers

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Abstract: The present theoretical paper treats the problem of determination of the coefficient of thermal expansion of a rod with two concentric longitudinal layers. The layers exhibit non-linear elastic behaviour. Besides, the layers have different thickness. The rod is exposed to temperature influence. The coefficient of thermal expansion of the rod is derived by analysing the thermal strains in the two layers. Since the layers are connected, the strains in the layers are equal (this fact is used to work out an equation for determining the coefficient of thermal expansion of the rod). The case when the two layers are continuously inhomogeneous along the rod length is also considered. In this case the material properties of the layers vary continuously in longitudinal direction. Here again the layers have non-linear elastic mechanical behaviour. Analysis of the distribution of strains along the length of the rod is carried-out when deriving the coefficient of thermal expansion of the rod. A comparison with the coefficient of thermal expansion of a rod having linear-elastic mechanical behaviour is performed for check-up of the solutions obtained in the present paper.

Keywords: Concentric layers, Rod, Circular section, Coefficient of thermal expansion

Introduction

Layered material systems find wide application in different sectors of modern technics (components of load-carrying structures, electronics, optical devices, sport facilities, engineering infrastructure, aeronautics, aerospace, car industry, etc.) (Dolgov, 2005; Kim et al., 1999; Nguyen et al., 2015). The increased usage of layered material systems in recent decades can be explained mainly by their excellent properties like high strength-to-weight and stiffness-to-weight ratios and very good processability (Rzhanitsyn, 1986; Rizov, 2017). The layered material systems represent combinations of layers made of different engineering materials (Kaul, 2014; Lloyd & Molina Aldareguia, 2003; Rizov, 2018). This fact is a premise for developing of diverse systems which combine in a highly efficient manner the advantages of the constituent materials. Besides, the layers may have different thickness.

In many practical applications in various areas of contemporary engineering layered components are subjected to temperature influence during their life-time. Besides, some layered material systems have non-linear elastic mechanical behaviour, i.e. their stress-strain constitutive law is non-linear. In such cases the Hook's law is not applicable. Also, the coefficients of thermal expansion of the layers are different. Besides, the layers may be manufactured by using inhomogeneous materials like functionally graded materials (the latter represent advanced continuously inhomogeneous composites (Gururaja Udupa et al., 2014; Radhika et al., 2020; Rizov, 2020; Toudehdehghan et al., 2017). In cases when layers are inhomogeneous, the material properties of a layer including the coefficient of thermal expansion change continuously along one or more directions.

The purpose of the present paper is to develop an analytical approach for determination of the coefficient of thermal expansion of a rod with two concentric longitudinal layers which have non-linear elastic mechanical behaviour. The moduli of elasticity and the coefficients of thermal expansion of layers are different. First, the case of homogeneous layers is treated. Then a solution is obtained also for a rod whose layers are continuously

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inhomogeneous in longitudinal direction. Comparisons with known solutions for the coefficient of thermal expansion are performed for check-up of the solutions derived in this paper.

Coefficient of Thermal Expansion Determination

Consider a rod of circular cross-section with radius, R. The rod is made of two concentric longitudinal layers. The radius of cross-section of the internal layer is r as shown in Fig. 1. The rod is subjected to uniform heating by temperature, Δt. The coefficients of thermal expansion of layers 1 and 2 of the rod are denoted by α_{t1} and α_{t2}, respectively. The layers are homogeneous.

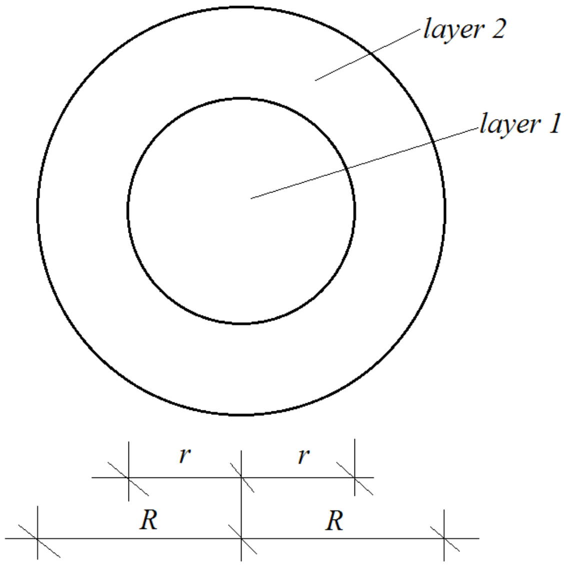

Figure 1. Cross-section of the rod.

The mechanical behaviour of the layers is non-linear elastic. The non-linear constitutive law of layer 1 is expressed by formula (1) (Lukash, 1997), i.e.

$$\sigma_1 = \frac{E_1 \varepsilon}{\sqrt{1 + s_1 \varepsilon^2}}, \quad (1)$$

where σ_1 is the stress, ε is the strain, E_1 is the modulus of elasticity, s_1 is a material property of this layer. Similarly, for the constitutive law of layer 2 of the rod we have

$$\sigma_2 = \frac{E_2 \varepsilon}{\sqrt{1 + s_2 \varepsilon^2}}. \quad (2)$$

Here σ_2 is the stress, ε is the strain, E_2 is the modulus of elasticity, s_2 is a material property. The purpose of the present analysis is to derive the coefficient of thermal expansion, α_{tc}, of the rod. As known, the prolongation, Δl_t, of the rod due to heating can be calculated by applying formula (3), i.e.

$$\Delta l_t = \alpha_{tc} \, \Delta t \, l, \quad (3)$$

where l is length of the rod.

The following approach is used here to derive α_{tc}. If the layers of the rod are not connected their prolongation can calculated by formulas (4) and (5), i.e.

$$\Delta l_{t1} = \alpha_{t1}\, \Delta t\, l\,, \tag{4}$$

$$\Delta l_{t2} = \alpha_{t2}\, \Delta t\, l\,, \tag{5}$$

where α_{t1} and α_{t2} are the coefficients of thermal expansion of layers 1 and 2, respectively. However, the rod layers are connected. As a result of this the prolongations of the layers are equal. This induces axial forces in layers 1 and 2 denoted by N_{t1} and N_{t2}, respectively. The prolongations, Δl_{Nt1} and Δl_{Nt2}, of the layers due to axial forces are found by formulas (6) and (7), i.e.

$$\Delta l_{Nt1} = \varepsilon_1 l\,, \tag{6}$$

$$\Delta l_{Nt2} = \varepsilon_2 l\,, \tag{7}$$

where the strains, ε_1 and ε_2, are found as

$$\varepsilon_1 = \frac{\sigma_1}{\sqrt{E_1^2 - s_1\sigma_1^2}}\,, \tag{8}$$

$$\varepsilon_2 = \frac{\sigma_2}{\sqrt{E_2^2 - s_2\sigma_2^2}}\,. \tag{9}$$

Formulas (8) and (9) are obtained by solving (1) and (2) with respect to strain. By using formulas (4) – (9), the prolongations of layers 1 and 2 of the rod due to heating and to axial forces are written as

$$\Delta l_1 = \Delta l_{t1} + \Delta l_{Nt1} = \alpha_{t1}\Delta t l + \frac{l\sigma_1}{\sqrt{E_1^2 - s_1\sigma_1^2}}\,, \tag{10}$$

$$\Delta l_2 = \Delta l_{t2} + \Delta l_{Nt2} = \alpha_{t2}\Delta t l + \frac{l\sigma_2}{\sqrt{E_2^2 - s_2\sigma_2^2}}\,. \tag{11}$$

Since the layers of the rod are connected, their prolongations are equal, i.e.

$$\Delta l_1 = \Delta l_2\,. \tag{12}$$

From (10), (11) and (12), we obtain

$$\alpha_{t1}\Delta t l + \frac{l\sigma_1}{\sqrt{E_1^2 - s_1\sigma_1^2}} = \alpha_{t2}\Delta t l + \frac{l\sigma_2}{\sqrt{E_2^2 - s_2\sigma_2^2}}\,. \tag{13}$$

The stresses, σ_1 and σ_2, in equation (13) are unknowns. One complementary equation is written by considering the equilibrium of the axial forces, N_{t1} and N_{t2}, in the layers of the rod, i.e.

$$N_{t1} = N_{t2}\,. \tag{14}$$

Since

$$N_{t1} = \sigma_1 A_1 \tag{15}$$

and

$$N_{t2} = \sigma_2 A_2 \tag{16}$$

formula (14) takes the following form

$$\sigma_1 A_1 = \sigma_2 A_2, \tag{17}$$

where the areas, A_1 and A_2, of the cross-sections of layers 1 and 2 are found as

$$A_1 = \pi r^2, \tag{18}$$

$$A_2 = \pi(R^2 - r^2). \tag{19}$$

The stresses are derived from equations (13) and (17) by MatLab. In order to determine α_{tc} we equalize Δl and Δl_{t1}, i.e.

$$\Delta l = \Delta l_{t1}. \tag{20}$$

Finally, by inserting of (3) and (10) in (20), we obtain the following equation with unknown, α_{tc}:

$$\alpha_{tc} \Delta t\, l = \alpha_{t1} \Delta t\, l + \frac{l \sigma_1}{\sqrt{E_1^2 - s_1 \sigma_1^2}}. \tag{21}$$

Equation (21) is solved with respect to α_{tc}, i.e.

$$\alpha_{tc} = \alpha_{t1} + \frac{\sigma_1}{\Delta t \sqrt{E_1^2 - s_1 \sigma_1^2}}. \tag{22}$$

Expression (22) is checked-up in the following way. It is clear from formulas (1) and (2) that at $s_1 = s_2 = 0$ the non-linear constitutive laws transform into the Hook's law, i.e.

$$\sigma_1 = E_1 \varepsilon, \tag{23}$$

$$\sigma_2 = E_2 \varepsilon. \tag{24}$$

This fact indicates that at $s_1 = s_2 = 0$ formula (22) for the coefficient of thermal expansion of the non-linear elastic rod derived in this paper should transform into the formula for the coefficient of thermal expansion of the linear-elastic rod. We are going to check-up this. For this purpose, we substitute $s_1 = s_2 = 0$ in (13) and (22) and by carrying-out some mathematical transformations we derive

$$\alpha_{tc} = \frac{\alpha_{t1} E_1 A_1 + \alpha_{t2} E_2 A_2}{E_1 A_1 + E_2 A_2}. \tag{25}$$

By inserting of (18) and (19) in (25), we have

$$\alpha_{tc} = \frac{\alpha_{t1} E_1 r^2 + \alpha_{t2} E_2 \left(R^2 - r^2\right)}{E_1 r^2 + E_2 \left(R^2 - r^2\right)}. \tag{26}$$

The fact that (26) coincides with the expression for the coefficient of thermal expansion of a linear-elastic rod published in (Hoa et al., 2003) is a check-up of (22).

The case of a non-linear elastic rod with two concentric layers which are continuously inhomogeneous in longitudinal direction is also considered. The rod cross-section is shown in Fig. 1. The moduli of elasticity and the coefficients of thermal expansion of the two layers change continuously along the rod length, i.e.

$$E_1 = E_1(x), \tag{27}$$

$$E_2 = E_2(x), \tag{28}$$

$$\alpha_{t1} = \alpha_{t1}(x), \tag{29}$$

$$\alpha_{t2} = \alpha_{t2}(x), \tag{30}$$

where

$$0 \le x \le l. \tag{31}$$

In formulas (27) – (30) x is the longitudinal centroidal axis of the rod. The prolongations of the two layers due to heating are written as

$$\Delta l_{t1} = \int_0^l \alpha_{t1}\, \Delta t\, dx, \tag{32}$$

$$\Delta l_{t2} = \int_0^l \alpha_{t2}\, \Delta t\, dx. \tag{33}$$

The prolongations of the two layers induced by the axial forces are expressed as

$$\Delta l_{Nt1} = \int_0^l \varepsilon_1 dx, \tag{34}$$

$$\Delta l_{Nt2} = \int_0^l \varepsilon_2 dx. \tag{35}$$

By combing of (8), (9), (32), (33), (34) and (35) we derive the following expressions for the prolongations of layers 1 and 2 of the continuously inhomogeneous rod due to heating and to axial forces:

$$\Delta l_1 = \int_0^l \alpha_{t1} \Delta t dx + \int_0^l \frac{\sigma_1}{\sqrt{E_1^2 - s_1 \sigma_1^2}} dx, \tag{36}$$

$$\Delta l_2 = \int_0^l \alpha_{t2} \Delta t dx + \int_0^l \frac{\sigma_2}{\sqrt{E_2^2 - s_2 \sigma_2^2}} dx. \tag{37}$$

The prolongations of the layers are equal, i.e.

$$\int_0^l \alpha_{t1} \Delta t dx + \int_0^l \frac{\sigma_1}{\sqrt{E_1^2 - s_1 \sigma_1^2}} dx = \int_0^l \alpha_{t2} \Delta t dx + \int_0^l \frac{\sigma_2}{\sqrt{E_2^2 - s_2 \sigma_2^2}} dx. \tag{38}$$

Equations (17) and (38) are solved with respect to stresses, σ_1 and σ_2.
By equalizing of (3) and (36) we have

$$\alpha_{tc}\,\Delta t\,l = \int_0^l \alpha_{t1}\Delta t dx + \int_0^l \frac{\sigma_1}{\sqrt{E_1^2 - s_1\sigma_1^2}}dx\,. \tag{39}$$

From equation (39) we determine

$$\alpha_{tc} = \frac{1}{\Delta t\,l}\left(\int_0^l \alpha_{t1}\Delta t dx + \int_0^l \frac{\sigma_1}{\sqrt{E_1^2 - s_1\sigma_1^2}}dx\right). \tag{40}$$

It should be mentioned that when α_{t1} and E_1 are constants (i.e., α_{t1} and E_1 do not change along the length of the rod) formula (40) transforms into expression (25) for the coefficient of thermal expansion of a non-linear elastic rod with two concentric layers which are homogeneous in longitudinal direction.

Conclusion

The problem for determination of the coefficient of thermal expansion of a rod with two concentric layers is considered theoretically. The rod has non-linear elastic behaviour that is treated by using a smooth stress-strain relationship. First, the case of a rod with two homogeneous layers is analyzed. Equations for determination of the coefficient of thermal expansion are worked out by analyzing the strains in the rod and by considering the equilibrium of axial forces in the two layers. The expression for the coefficient of thermal expansion obtained in the present paper is checked-up by comparing with a known formula for the coefficient of thermal expansion of a rod having linear-elastic behaviour (it is shown that the expression derived transforms into the known formula by substituting of $s_1 = s_2 = 0$ in the equations for obtaining the coefficient of thermal expansion. The problem of determination of the coefficient of thermal expansion of a non-linear elastic rod with two concentric layers which are made of materials that are continuously inhomogeneous in longitudinal direction is also treated. The coefficients of thermal expansions determined in the present paper can be applied for calculating the prolongations of non-linear elastic rods under heating. The equations worked out can also be used for analyzing the stressed and strained state of the rod layers due to heating. It should be noted that the approach for determining the coefficient of thermal expansion presented in this paper can be developed further by considering various non-linear stress-strain relationships and laws for distributions of material properties of the inhomogeneous layers in longitudinal direction.

Recommendations

The theoretical approach presented in this paper can be used for determination of the coefficient of thermal expansion of rods with two concentric layers having non-linear elastic behavior.

Scientific Ethics Declaration

The author declares that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the author.

Acknowledgements or Notes

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References

Dolgov, N. (2005). Determination of stresses in a two-layer coating. *Strength of Materials*, *37*, *422-431*.

Hoa, S.V., Gay, D., & Tsai, S W. (2003). *Composite materials* (1st ed.). Boca Raton: CRC Press.

Kaul, A. B. (2014). Two-dimensional layered materials: Structure, properties, and prospects for device applications, *Journal of Material Research*, *29*, 348-361.

Kim, J. S., Paik, K.W., & Oh, S. H. (1999). The multilayer-modified Stoney's formula for laminated polymer composites on a silicon substrate. *Journal of Applied Physics*, *86*, 5474–5479.

Lloyd, S.J., & Molina-Aldareguia, J. M. (2003), Multilayered materials: a palette for the materials artist. *Phil. Transactions R. Soc. Lond. A*, *361*(1813), 2931-2940.

Lukash, P. (1997). *Fundamentals of non-linear structural mechanics*. Science.

Nguyen, S. N., Lee, J., & Cho, M. (2015). Efficient higher-order zig-zag theory for viscoelastic laminated composite plates. *International Journal of Solids and Structures*, *62*, *174-185*.

Radhika, N., Sasikumar, J., Sylesh, J. L., & Kishore, R. (2020). Dry reciprocating wear and frictional behaviour of B4C reinforced functionally graded and homogenous aluminium matrix composites. *Journal of Materials Research and Technology*, *9*(2), *1578-1592*.

Rizov, V. I. (2020). Longitudinal vertical crack analysis in beam with relaxation stresses. *World Journal of Engineering*, *18*(3), 452-457.

Rizov, V.I. (2017). Analysis of longitudinal cracked two-dimensional functionally graded beams exhibiting material non-linearity. *Frattura ed Integrita Strutturale*, *11*(41), *498-510*.

Rizov, V.I. (2018). Analysis of cylindrical delamination cracks in multilayered functionally graded non-linear elastic circular shafts under combined loads. *Frattura ed Integrità Strutturale*, *12*(46), 158-17.

Rzhanitsyn, A.R. (1986). *Built-up bars and plates*. Stroyiizdat.

Toudehdehghan, J., Lim, W., Foo1, K. E., Ma'arof, M. I. N., & Mathews J. (2017). A brief review of functionally graded materials. *MATEC Web of Conferences*, *131*.

Udupa, G., Rao, S. S., & Gangadharan, K. V. (2014). Functionally graded composite materials: An overview. *Procedia Materals Science*, *5*, 1291-1299.

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