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Approximate Analytic Solution for Fractional Differential Equations with a Generalized Fractional Derivative of Caputo-Type

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Abstract: This paper introduces the analytic series solution of the differential equation with fractional Caputo-type derivative including two parameters using the homotopy analysis method (HAM). The main properties of the fractional derivative with two parameters are illustrated. The standard HAM converges for a short domain, so we modify the method to overcome this issue by dividing the domain into finite subintervals and applying the method to each one. The initial conditions in each subinterval can be obtained from the previous one In this way, a continuous piecewise function that converges to the exact solution can be constructed. The effect of each fractional parameter on the solution behaviors is presented in figures and tables. Several examples are presented to verify the validity of the algorithm. A comparison with the exact solution in the case of integer derivative and with the Adaptive predictor corrected algorithm in the case of fractional one demonstrates the efficiency of the method.

Keywords: Fractional calculus, Homotopy analysis method, Riccati equation.

Introduction

Many different derivatives have been introduced, such as Hadamard, Grunwald Letnikov, Riesz, Caputo, and more (Kilbas et al., 2006; Samko et al., 1993). Based on the fractional integral, the fractional derivative is defined. One of these fractional integrals is the Riemann-Liouville fractional integral of order $\alpha > 0$, which is defined by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t > a, \tag{1}$$

This definition is widely studied. If we are based on this definition, the Riemann-Liouville fractional derivative and the Caputo-type fractional derivative of order $\alpha > 0$, are defined by

$$D_{a+}^\alpha (f(t)) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-\alpha-1} f(s) ds \tag{2}$$

$$D_{a+}^\alpha (f(t)) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds \tag{3}$$

respectively, where $m-1 < \alpha \leq m$, and $m \in \aleph$. The Caputo fractional operator is widely used to model many physical problems in fractional calculus applications because it is suitable for initial value problems and has many characteristics similar to ordinary derivatives. For example, when $m-1 < \alpha \leq m$, the Caputo operator satisfies the rule

$$I_{a+}^{\alpha}D_{a+}^{\alpha}(f(t)) = f(t) - \sum_{n=0}^{m-1} \frac{f^{(k)}(a)}{n!}(t-a)^n, \quad t > a. \tag{4}$$

Recently, (Odibat & Baleanu, 2020) introduced the generalized Caputo-type fractional derivative with properties similar to those of the Caputo derivative. This type of generalized derivative looks more like ordinary derivatives than other generalized derivatives. The generalized fractional derivative of a continuous function f is defined by

$$D_a^{\alpha,\rho}(f(t)) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{m-\alpha-1}(s^{1-\rho}\frac{d}{ds})^m f(s)ds, \quad t > a, \tag{5}$$

this generalized order $\alpha > 0$, and $\rho > 0$, where $a \geq 0$ and $m-1 < \alpha \leq m$. Until now, there is no algorithm to solve fractional differential equations in the sense of a generalized fractional derivative of the Caputo-type by approximate analytic technique. Moreover, is the solution valid for a long time span? In this paper, we applied our result to the Riccati equation which is used in different areas of mathematics, such as in algebraic geometry the theory of conformal mapping, and physics. It also appears in many applications in engineering and science domains, such as robust stabilization, stochastic realization theory, network synthesis, optimal control, and financial mathematics. Recently, many researchers studied the Riccati differential equation of fractional order, such as (Cang et al., 2009) used HAM to solve non-linear Riccati differential equation with fractional order. The fractional Riccati equation with the generalized fractional derivative of Caputo-type definition can be written as:

$$D_a^{\alpha,\rho}y(t) = Ay(t) + By^2(t) + C,$$

subject to the initial condition $y(a) = c$. In the case of $\alpha = \rho = 1$, the fractional equation reduces to the classical Riccati differential equation.

Preliminaries

Recently, Almeida et al.[6] have introduced the Caputo–Katugampola derivative with two parameters defined by

$$D_{a+}^{\alpha,\rho}(f(t)) = \frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_a^t (t^\rho - s^\rho)^{-\alpha}f'(s)ds, 0 < \alpha \leq 1, t > a \geq 0. \tag{6}$$

Later, (Odibat & Baleanu, 2020) extended the definition of two parameters, $m-1 < \alpha \leq m$ in arbitrary order α and $\rho > 0$. The generalized fractional derivative of a continuous function f is $D_a^{\alpha,\rho}(f(t))$ of order $\alpha > 0$ and $\rho > 0$ can be written as

$$D_a^{\alpha,\rho}(f(t)) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{m-\alpha-1}(s^{1-\rho}\frac{d}{ds})^m f(s)ds, \quad t > a, \tag{7}$$

where $a \geq 0$ and $m-1 < \alpha \leq m$. We note that as $\rho = 1$ in (7), we have the standard Caputo fractional derivative in (3).

Definition 1 (Odibat & Baleanu, 2020) *The generalized fractional integral of the function* f, $I_{0+}^{\alpha,\rho}(f(t))$, *of order* $0 < \alpha$, *where* $\rho > 0$, *is defined by:*

$$I_{a+}^{\alpha,\rho}(f(t)) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{a+}^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}f(s)ds. \tag{8}$$

The following property gives us the relation between generalized fractional integral and generalized Caputo-type fractional derivative(Odibat & Baleanu, 2020)] If $m-1 < \alpha \leq m$, $a \geqslant 0$, $\rho > 0$ and $f \in C^m[a,b]$, then for $a < t \leq b$,

$$I_{a+}^{\alpha,\rho}D_{a+}^{\alpha,\rho}(f(t)) = f(t) - \sum_{n=0}^{m-1} \frac{1}{\rho^n n!}(t^\rho - a^\rho)^n \left[\left(x^{1-\rho}\frac{d}{dx}\right)^n f(x)\right]|_{x=a}. \tag{9}$$

In the following, we present some propositions and basic properties of generalized fractional derivatives and integrals with two parameters, which we will need in the next chapter.

Proposition 1 *Let* $m-1<\alpha\leq m$, $\rho>0$, *and* $n\in\mathbb{R}$ *then we have:*

$$I_{0+}^{\alpha,\rho}(t^n)=\frac{\rho^{-\alpha}t^{(\rho\alpha+n)}\Gamma(\frac{n}{\rho}+1)}{\Gamma(\alpha+1+\frac{n}{\rho})},\quad \Re(\rho)>0,\Re(n+\rho)>0,\Re(\alpha)>0. \tag{10}$$

proof: Let $n\in\mathbb{R}$, and $\alpha,\rho>0$ then we have

$$I_{0+}^{\alpha,\rho}(t^n)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha-1}s^n ds, \tag{11}$$

$$=\frac{\rho^{1-\alpha}t^{\rho(\alpha-1)}}{\Gamma(\alpha)}\int_0^t s^{\rho-1+n}\left(1-\left(\frac{s}{t}\right)^\rho\right)^{\alpha-1}ds, \tag{12}$$

Set $u=(\frac{s}{t})^\rho$, so $s=u^{\frac{1}{\rho}}t$, and when $s=0$, $u=0$, when $s=t$, $u=1$, then Eq.(12) becomes

$$=\frac{\rho^{-\alpha}t^{(\rho\alpha+n)}}{\Gamma(\alpha)}\int_0^1 u^{\frac{u}{\rho}+1-1}(1-u)^{\alpha-1}du,$$

So, $I_{0+}^{\alpha,\rho}(t^n)=\frac{\rho^{-\alpha}t^{(\rho\alpha+n)}\Gamma(\frac{n}{\rho}+1)}{\Gamma(\alpha+1+\frac{n}{\rho})}$.

Remark1:

$$I_{0+}^{\alpha,\rho}(t)=\frac{\rho^{-1-\alpha}t^{(\rho\alpha+1)}\Gamma(\frac{1}{\rho})}{\Gamma(\alpha+1+\frac{1}{\rho})},\quad \Re(\rho)>0,\Re(\alpha)>0. \tag{13}$$

$$I_{0+}^{\alpha,\rho}(C)=\frac{C\rho^{-\alpha}t^{(\rho\alpha)}}{\alpha\Gamma(\alpha)},\quad \Re(\rho)>0,\Re(\alpha)>0,\quad \text{C is constant.} \tag{14}$$

If $m-1<\alpha\leq m$, $a\geqslant 0$, $\rho>0$, $C\in\mathbb{R}$ and $f,g\in C^m[a,b]$, then for $a<t\leq b$, we have:

$$I_{0+}^{\alpha,\rho}(Cf(t)+g(t))=CI_{0+}^{\alpha,\rho}(f(t))+I_{0+}^{\alpha,\rho}(g(t)).$$

Proposition 2 *Let* $0<\alpha\leq m$, $\rho>0$, *and* $n\in\mathbb{R}$ *then we have:*

$$D_{0+}^{\alpha,\rho}(t^n)=\frac{\rho^{\alpha-m}t^{n-\alpha\rho}\Gamma(n+1)}{\Gamma(n-m+1)}\frac{\Gamma(\frac{n}{\rho}-m+1)}{\Gamma(\frac{n}{\rho}-\alpha+1)},\quad \Re(m)>\Re(\alpha). \tag{15}$$

Proof: Let $n\in\mathbb{R}$, and $\alpha,\rho>0$ then we have

$$D_{0+}^{\alpha,\rho}(t^n)=\frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{m-\alpha-1}\left(s^{1-\rho}\frac{d}{ds}\right)^m s^n ds, \tag{16}$$

$$=\frac{\rho^{\alpha-m+1}\Gamma(n+1)t^{\rho(m-\alpha-1)}}{\Gamma(m-\alpha)\Gamma(n-m+1)}\int_0^t s^{\rho-1-\rho m+n}\left(1-(\frac{s}{t})^\rho\right)^{m-\alpha-1}ds \tag{17}$$

Set $u=(\frac{s}{t})^\rho$, so $s=u^{\frac{1}{\rho}}t$, and when $s=0$, $u=0$, when $s=t$, $u=1$, then Eq.(17) becomes

$$D_{0+}^{\alpha,\rho}(t^n)=\frac{\rho^{\alpha-m}\Gamma(n+1)t^{n-\alpha\rho}}{\Gamma(m-\alpha)\Gamma(n-m+1)}\int_0^1 u^{\frac{n}{\rho}-m}(1-u)^{m-\alpha-1}du,\qquad \text{where}\quad \Re(n+\rho)>\Re(\rho m),$$

$$=\frac{\rho^{\alpha-m}\Gamma(n+1)t^{n-\alpha\rho}}{\Gamma(m-\alpha)\Gamma(n-m+1)}\beta\left(\frac{n}{\rho}-m+1,m-\alpha\right)$$

That is $D_{0+}^{\alpha,\rho}(t^n)=\frac{\rho^{\alpha-m}t^{n-\alpha\rho}\Gamma(n+1)}{\Gamma(n-m+1)}\frac{\Gamma(\frac{n}{\rho}-m+1)}{\Gamma(\frac{n}{\rho}-\alpha+1)}$.

Remark2:

$$D_{0+}^{\alpha,\rho}(t)=\frac{\rho^{\alpha-m}t^{1-\alpha\rho}\Gamma(1-m+\frac{1}{\rho})}{\Gamma(1-\alpha+\frac{1}{\rho})},\quad \Re(\rho)+1>\Re(\rho m),\Re(m)>\Re(\alpha). \tag{18}$$

$$D_{0+}^{\alpha,\rho}(C)=0,\quad \mathrm{C}\quad \text{is}\quad \text{constant}. \tag{19}$$

Proposition 3 *Let* $m-1<\alpha\le m$, $\rho>0$, *and* $n\in\mathbb{R}$, *then we have:*

$$I_{a+}^{\alpha,\rho}(t^n)=\frac{\rho^{-\alpha}\Gamma(\frac{n}{\rho}+1)}{\Gamma(\frac{n}{\rho}+1+\alpha)}(t^{\alpha\rho+n}-a^{n+\alpha\rho}). \tag{20}$$

Proof: Let $n\in\mathbb{R}$, $a>0$, and $\alpha,\rho>0$. Then we have

$$I_{a+}^{\alpha,\rho}((t)^n)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1+n}(t^\rho-s^\rho)^{\alpha-1}ds, \tag{21}$$

$$=\frac{\rho^{1-\alpha}t^{\rho(\alpha-1)}}{\Gamma(\alpha)}\int_a^t s^{\rho-1+n}\left(1-\left(\frac{s}{t}\right)^\rho\right)^{\alpha-1}ds, \tag{22}$$

set $u=(\frac{s}{t})^\rho$, so $s=u^{\frac{1}{\rho}}t$, and when $s=a$, $u=(\frac{a}{t})^\rho$, when $s=t$, $u=1$, then Eq.(22) becomes

$$I_{0+}^{\alpha,\rho}(t^n)=\frac{\rho^{-\alpha}t^{(\rho\alpha+n)}}{\Gamma(\alpha)}\left(\int_{(\frac{a}{t})^\rho}^{0} u^{\frac{n}{\rho}}(1-u)^{\alpha-1}du+\int_0^1 u^{\frac{n}{\rho}}(1-u)^{\alpha-1}du\right),$$

Thus, $$I_{a+}^{\alpha,\rho}(t^n)=\frac{\rho^{-\alpha}\Gamma(\frac{n}{\rho}+1)}{\Gamma(\alpha+1+\frac{n}{\rho})}(t^{\rho\alpha+n}-a^{n+\alpha\rho}).$$

Remark4:

$$I_{a+}^{\alpha,\rho}(t)=\frac{\rho^{-1-\alpha}\Gamma(\frac{1}{\rho})}{\Gamma(\frac{1}{\rho}+1+\alpha)}(t^{\alpha\rho+1}-a^{1+\alpha\rho}). \tag{23}$$

$$I_{a+}^{\alpha,\rho}(C)=\frac{C\rho^{-\alpha}}{\alpha\Gamma(\alpha)}(t^{\alpha\rho}-a^{\alpha\rho}),\mathrm{C}\quad \text{is}\quad \text{constant}. \tag{24}$$

Proposition 4 *Let* $m-1<\alpha\le m$, $\rho>0$, *and* $n\in\mathbb{R}$ *then we have:*

$$I_{a+}^{\alpha,\rho}((t^\rho-a^\rho)^n)=\frac{\rho^{-\alpha}(t^\rho-a^\rho)^{n+\alpha}\Gamma(n+1)}{\Gamma(\alpha+n+1)}. \tag{25}$$

Proof: $I_{a+}^{\alpha,\rho}((t^\rho-a^\rho)^n)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha-1}(s^\rho-a^\rho)^n ds$

Set $z=s^\rho$, so $s=z^{\frac{1}{\rho}}$, and when $s=a$, $z=a^\rho$, when $s=t$, $z=t^\rho$, then

$$=\frac{\rho^{-\alpha}}{\Gamma(\alpha)}\int_{a^\rho}^{t^\rho}(t^\rho-z)^{\alpha-1}(z-a^\rho)^n dz$$

Next, we take $r=\frac{z-a^\rho}{t^\rho-a^\rho}$, so $z=a^\rho+r(t^\rho-a^\rho)$ and $\mathrm{d}r=\frac{\mathrm{d}z}{t^\rho-a^\rho}$, and when $z=a^\rho$, $r=0$, when $z=t^\rho$, $r=1$, then we get

$$=\frac{\rho^{-\alpha}}{\Gamma(\alpha)}(t^\rho-a^\rho)^{n+\alpha}\int_0^1(1-r)^{\alpha-1}r^n dr,$$

That is $I_{a+}^{\alpha,\rho}((t^\rho-a^\rho)^n)=\frac{\rho^{-\alpha}(t^\rho-a^\rho)^{n+\alpha}\Gamma(n+1)}{\Gamma(\alpha+n+1)}$.

Proposition 5 (Odibat & Baleanu, 2020) *If* $m-1<\alpha\le m$, $n>m-1$ *and* $n\notin\aleph$, *then*

$$D_{a+}^{\alpha,\rho}(t^\rho - a^\rho)^n = \frac{\rho^\alpha (t^\rho - a^\rho)^{n-\alpha}\Gamma(n+1)}{\Gamma(n-\alpha+1)}. \quad (26)$$

The Homotopy Analysis Method

In this work, we consider the following IVP

$$D_{0+}^{\alpha,\rho} y(t) = \mathcal{N}[y,t], \quad (27)$$

where $\mathcal{N}$ is the nonlinear and $y(t)$ is an unknown function. The zeroth-order deformation equation can be defined as

$$(1-q)\mathcal{L}[\phi(t;q) - y_0(t)] + q\hbar(D_{0+}^{\alpha,\rho}[\phi(t;q)] - \mathcal{N}[\phi(t;q)]) = 0, \quad (28)$$

where q is an embedding parameter belonging to $[0,1]$, $\hbar$ is the convergence control parameter, $\mathcal{L}$ is an auxiliary linear operator, $\mathcal{N}$ is the nonlinear operator and $y_0(t)$ is an initial guess of the exact solution $y(t)$. Now, we can expand $\phi(t;q)$ into a Taylor series with respect to q:

$$\phi(t;q) = y_0(t) + \sum_{j=1}^{\infty} y_j(t) q^j. \quad (29)$$

When the embedding parameter q increases from 0 to 1; $\phi(t;0) = y_0(t)$, $\phi(t;1) = y(t)$, the solution $\phi(t;q)$ of the zeroth-order deformation (28) varies continuously from the initial guess $y_0(t)$ to the exact solution $y(t)$. If the auxiliary linear operator, the initial guess $y_0(t)$, and $\hbar$, are appropriately chosen, that (29) converges at $q = 1$, then we have the so-called homotopy-series solution

$$y(t) = y_0(t) + \sum_{j=1}^{\infty} y_j(t). \quad (30)$$

By differentiating equation (28) i-time with respect to q and then putting $q = 0$, and finally dividing them by $i!$, we have the i-th order deformation equation:

$$\mathcal{L}[y_i(t) - \chi_i y_{i-1}(t)] = \hbar[D_{0+}^{\alpha,\rho} y_{i-1}(t) - \mathcal{R}_i(\vec{y}_{i-1}(t))], \quad (31)$$

where $\mathcal{R}_i(\vec{y}_{i-1}(t)) = \frac{1}{i!}\frac{\partial^i[\mathcal{N}[\phi(t;q)]]}{\partial q^i}, \quad \chi_i = \begin{cases} 0, & i \leq 1 \\ 1, & i > 1 \end{cases},$

and $\vec{y}_{i-1}(t)$ is a vector $[y_o(t), \quad y_1(t), \quad y_2(t), \ldots\ldots, \quad y_{i-1}(t)]$. By taking $\mathcal{L}^{-1}$ for Eq (31), we get:

$$y_i(t) = \chi_i y_{i-1}(t) + \hbar\mathcal{L}^{-1}[D_{0+}^{\alpha,\rho} y_{i-1}(t) - \mathcal{R}_i(\vec{y}_{i-1}(t))].$$

where $\mathcal{L}^{-1}$ is the inverse of the linear operator. The solution can be expressed as:

$$y(t) = \sum_{j=0}^{M} y_j(t). \quad (32)$$

Consider the fractional equation:

$$D_{0+}^{\alpha,\rho}(y(t)) = \mathcal{N}[y,t], \quad (33)$$

subject to the initial condition $y(0) = c$, where $D_{a+}^{\alpha,\rho}$ is the generalized Caputo-type fractional derivative operator of parameters α and ρ. From now the value of m in $D_{a+}^{\alpha,\rho}$ is one. By assuming the linear operator $\mathcal{L} = D_{0+}^{\alpha,\rho}$, Eq.(31) becomes

$$D_{0+}^{\alpha,\rho}[y_i(t) - \chi_i y_{i-1}(t)] = \hbar[D_{0+}^{\alpha,\rho} y_{i-1}(t) - \mathcal{R}_i(\vec{y}_{i-1}(t))]. \quad (34)$$

Now, apply $I_{0+}^{\alpha,\rho}$(10) on the right-hand and left-hand of Eq.(34), then we get

$$I_{0+}^{\alpha,\rho} D_{0+}^{\alpha,\rho}[y_i(t) - \chi_i y_{i-1}(t)] = \hbar[I_{0+}^{\alpha,\rho} D_{0+}^{\alpha,\rho} y_{i-1}(t) - I_{0+}^{\alpha,\rho}[\mathcal{R}_i(\vec{y}_{i-1}(t))]]. \quad (35)$$

Next step, apply property (9) on the left-hand, and the first part on the right-hand side, take $0 < \alpha \leq 1$, then Eq.(35) becomes

$$y_i(t) = (\chi_i + \hbar)y_{i-1}(t) + y_i(0) - \chi_i y_{i-1}(0) - \hbar y_{i-1}(0) - \hbar I_{0+}^{\alpha,\rho}[\mathcal{R}_i(\vec{y}_{i-1}(t))]. \qquad (36)$$

where the initial condition is $y_i(0) = \begin{cases} 0, & if \quad i = 1,2,3,.. \\ c, & if \quad i = 0. \end{cases}$

Multistage HAM

To express the solution by our proposed method, consider the generalized Caputo-type fractional IVP

$$D_{a+}^{\alpha,\rho}(y(t)) = f(t,y), \quad y(a) = c. \qquad (37)$$

where $0 < \alpha \leq 1$ and $0 \leq t \leq T$. According to the i-th order of the deformation equation

$$D_{a+}^{\alpha,\rho}[y_i(t) - \chi_i y_{i-1}(t)] = \hbar[D_{a+}^{\alpha,\rho} y_{i-1}(t) - \mathcal{R}_i(\vec{y}_{i-1}(t))], \qquad (38)$$

apply $I_{a+}^{\alpha,\rho}$(20) on Eq.(38) on the right-hand and left-hand sides:

$$I_{a+}^{\alpha,\rho} D_{a+}^{\alpha,\rho}[y_i(t) - \chi_i y_{i-1}(t)] = \hbar[I_{a+}^{\alpha,\rho} D_{a+}^{\alpha,\rho} y_{i-1}(t) - I_{a+}^{\alpha,\rho}\mathcal{R}_i(\vec{y}_{i-1}(t))],$$

then use the property (9), we have:

$$y_i(t) - \chi_i y_{i-1}(t) - y_i(a) + \chi_i y_{i-1}(a) = \hbar(y_{i-1}(t) - y_{i-1}(a)) - \hbar I_{a+}^{\alpha,\rho}(\mathcal{R}_i(\vec{y}_{i-1}(t))).$$

Thus, the i-th order of the approximate solution is given by:

$$y_i(t) = (\chi_i + \hbar)y_{i-1}(t) + y_i(a) - (\chi_i + \hbar)y_{i-1}(a) - \hbar I_{a+}^{\alpha,\rho}(\mathcal{R}_i(\vec{y}_{i-1}(t))). \qquad (39)$$

where the initial condition is $y_i(a) = \begin{cases} 0, & if \quad i = 1,2,3,.. \\ c, & if \quad i = 0. \end{cases}$ We can simplify(39) to becomes

$$y_i(t) = (\chi_i + \hbar)y_{i-1}(t) - (1 - \chi_i)(\chi_i + \hbar)y_{i-1}(t) - \hbar I_{a+}^{\alpha,\rho}(\mathcal{R}_i(\vec{y}_{i-1}(t))), \qquad (40)$$

So, the general solution of Eq.(37) using M-terms of the series is

$$Y(t) = \sum_{j=0}^{M} y_j(t). \qquad (41)$$

To evaluate the solution $Y(t)$ given in (41) along $[a, T]$, we present our technique in the following steps:

1. Divide the interval $[a, T]$ into N equal sub-intervals as

$[t_0, t_1], [t_1, t_2], \ldots\ldots, [t_n, t_{n+1}] \ldots, [t_{N-1}, t_N]$, where $\begin{cases} t_0 = a, \\ t_n = n\Delta t, \\ \Delta t = \frac{T-a}{N}, \\ t_N = T, \end{cases}$ and Δt is the step size.

2. The solution along the sub-interval $[t_0, t_1]$ is given by a substitute $t_0 = a = 0$, and $y(0) = c$, and we label it as $Y_0(t)$.
3. To ensure the continuity of the solution we replace $c = Y_0(t_1)$, and $t_1 = a$, so that we have $Y_1(t)$.
4. Using the same manner the solution along the sub-interval $[t_n, t_{n+1}]$ has $t_n = a + n\Delta t$, and $c = Y_{n-1}(t_n)$, which is labeled by $Y_n(t)$. Thus, our continuous solution can be written as

$$\tilde{Y}(t)=\begin{cases} Y_0(t) & : \quad a\le t\le t_1 \\ Y_1(t) & : \quad t_1\le t\le t_2 \\ Y_2(t) & : \quad t_2\le t\le t_3 \\ \vdots & \\ Y_n(t) & : \quad t_n\le t\le t_{n+1} \\ \vdots & \\ Y_N(t) & : \quad t_{N-1}\le t\le t_N=T \end{cases}$$

where $Y_n(t), n=0,1,2,\ldots,N$ is the continuous solution for (37) on the interval $[t_n,t_{n+1}]$, with I.C $y(t_n)=Y_{n-1}(t_n)$.

Numerical Results

Consider the following fractional Riccati equation

$$D_a^{\alpha,\rho}y(t)=2y(t)-y^2(t)+1, \tag{42}$$

$t>0,\ 0<\alpha\le 1$, subject to the initial condition $y(a)=c$, where $D_a^{\alpha,\rho}$ is the generalized Caputo-type fractional derivative operator, of parameters α and ρ. The exact solution of the Riccati Eq.(45), when $\alpha=1$ and $\rho=1$, is

$$y(t)=1+\sqrt{2}\tanh[\sqrt{2}t+\tfrac{1}{2}\log(\tfrac{\sqrt{2}-1}{\sqrt{2}+1})]. \tag{43}$$

According to MHAM for some $T>0$, the $i-th$ approximate solutions y_i is determined by the formula:

$$y_i(t)=(\chi_i+\hbar)y_{i-1}(t)+y_i(a)-(\chi_i+\hbar)y_{i-1}(a)-\hbar I_{a+}^{\alpha,\rho}(\mathcal{R}_i(\vec{y}_{i-1}(t))),$$

where $\quad \mathcal{R}_i(\vec{y}_{i-1}(t))=2y_{i-1}-\sum_{j=0}^{i-1}y_jy_{i-1-j}+(1-\chi_i)$.

The initial condition is $y_i(a)=\begin{cases}0, & if \quad i=1,2,3,.. \\ c, & if \quad i=0.\end{cases}$

For simplicity, let $\hbar=-1$. Using multistage HAM with uniform nodes, the first term of the series solution is:

$$y_1(t)=-\frac{\rho^{-\alpha}a^{\alpha\rho}}{\alpha\Gamma(\alpha)}+\frac{c^2\rho^{-\alpha}a^{\alpha\rho}}{\alpha\Gamma(\alpha)}-\frac{2c\rho^{-\alpha}a^{\alpha\rho}}{\alpha\Gamma(\alpha)}-\frac{c^2\rho^{-\alpha}t^{\alpha\rho}}{\alpha\Gamma(\alpha)}+\frac{2c\rho^{-\alpha}t^{\alpha\rho}}{\alpha\Gamma(\alpha)}+\frac{\rho^{-\alpha}t^{\alpha\rho}}{\alpha\Gamma(\alpha)}.$$

In the same way, we can find $y_i(t)$, for i=2,3,….The continuous solution of the Maclaurin series of (45), when $\alpha=1$, $\rho=1$, and $\Delta t=0.1$, can be written as:

$$\tilde{Y}(t)=$$
$$\begin{cases} Y_0(t)=\frac{t^3}{3}+t^2+t & : \quad 0\le t\le 0.1 \\ Y_1(t)=0.150869t^3+1.0299t^2+0.997987t+0.0000844281 & : \quad 0.1\le t\le 0.2 \\ Y_2(t)=-0.131482t^3+1.15935t^2+0.977604t+0.00124216 & : \quad 0.2\le t\le 0.3 \\ Y_3(t)=-0.491978t^3+1.43105t^2+0.90856t+0.00723595 & : \quad 0.3\le t\le 0.4 \\ \vdots & \\ Y_9(t)=-0.669366t^3+0.899396t^2+1.72969t-0.270041 & : \quad 0.9\le t\le 1 \end{cases}$$

In Table 1 we compare the approach with the adaptive P-C algorithm (Odibat & Baleanu, 2020) to fractional Riccati Eq.(42) when $\alpha=1$, and $\rho=1$ at $t=1$, $t=2$, and $t=5$. It's clear from our data in Table1, that the numerical solutions obtained using our algorithm completely agreed with the exact solution (y_{Exact}) more than the adaptive P-C algorithm. we can observe when the step size Δt decreases our approximate values of $y(t)$ converge to the exact solutions. Table 2, presents numerical solutions obtained using our algorithm at $t=2$, r, for different values of α and ρ. It is shown that the solution does not depend only on α but also on ρ. Figure 1 shows the exact solution with the approximate solutions of fractional Riccati Eq.(42) for $\alpha=\rho=1$, and $\Delta t=0.01$.

Table 1. Numerical solutions to fractional Riccati Eq.(45) when $\alpha = 1$ and $\rho = 1$.

	MHAM Algorithm			The Adptive P-C Algorithm		
Δt	t=1	t=2	t=5	t=1	t=2	t=5
1/10	1.68967243	2.35779846	2.41420179	1.68745117	2.35530727	2.41419835
1/40	1.68950123	2.35777211	2.41420167	1.68936339	2.35763805	2.41420152
1/160	1.68949843	2.35777166	2.41420167	1.68948986	2.35776357	2.41420166
1/640	1.68949839	2.35777165	2.41420167	1.68949786	2.35777115	2.41420167
y_Exact	1.68949839	2.35777165	2.41420167	1.68949839	2.35777165	2.41420167

Table 2. Numerical solutions to fractional Riccati Eq. (45) when $t = 2$.

	MHAM Algorithm			The Adptive P-C Algorithm		
Δt	α=1,ρ=0.9	α=0.95,ρ=0.75	α=0.9,ρ=1.2	α=1,ρ=0.9	α=0.95,ρ=0.75	α=0.9,ρ=1.2
1/10	2.36822637	2.32198326	2.22052899	2.36576348	2.34646084	2.26631061
1/40	2.36818244	2.32264980	2.21660104	2.36805246	2.34876916	2.26890814
1/160	2.36818165	2.32298363	2.21583356	2.36817385	2.34889710	2.26907235
1/640	2.36818164	2.32307541	2.21565525	2.36818116	2.34890540	2.26908393
1/1280	2.36818164	2.32309099	2.21562605	2.36818153	2.34890584	2.26908459

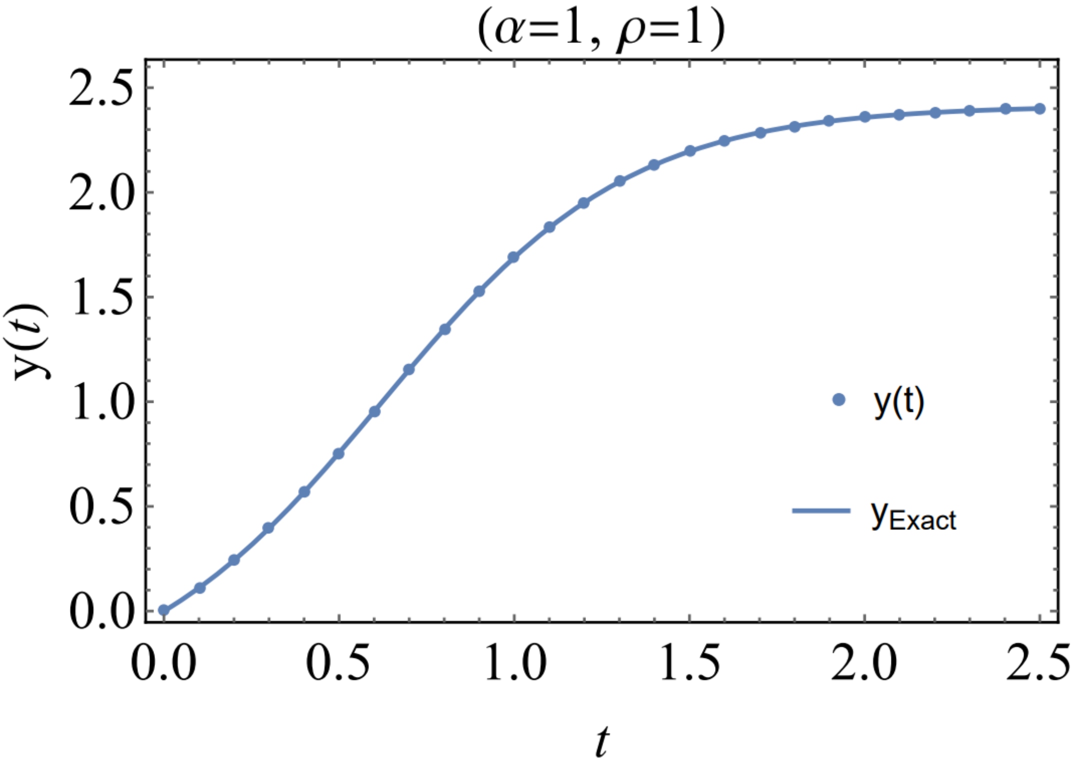

Figure 1. The approximation solutions and the Exact solution for (45), when $\Delta t = 0.01$, and $t \in [0,2.5]$.

Conclusion

This work successfully presented the analytic solution of the Riccati equation with a generalized Caputo fractional derivative in two parameters. A general framework for using the multistage homotopy analysis method for this kind of problem is constructed. Several properties of the fractional Caputo fractional derivative with two parameters are proved. The discussed example shows the efficiency of the new algorithm in terms of convergence and accuracy.

Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

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