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Some Fixed Point Results of $(\alpha,\beta,\varphi,\delta)$-Contractions in Extended Cone *b*-Metric Spaces

Zamir Selko
University of Elbasan “Aleksandër Xhuvani”

Eriola Sila
University of Tirana

Abstract This paper investigates the existence and uniqueness of fixed points for a class of generalized contractive mappings defined on extended cone metric spaces. Subsequently, we define and explore $(\alpha, \beta, \varphi, \delta)$-contractions, a generalization of traditional contractions that allows for a more nuanced understanding of the contraction behavior in extended cone metric spaces. The extended metric space was defined for the first time in 2017, by Kamran et al. (2017). They replaced the constant in the triangle inequality of the metric with a two-variable function and explored various fixed point theorems. In 2022, Das and Bag[4] introduced extended cone metric spaces by incorporating a three-variable map into the third condition of the cone metric. Afterwards, Selko and Sila introduced the concept of extended quasi cone b-metric spaces and demonstrated the Banach contraction within this framework. In 2018, Alqahtani. (2018) and colleagues established several fixed point results for a pair of orbital cyclic functions in an extended metric space. In this paper we prove some fixed point theorems for (α,β)-orbital-cyclic functions in extended cone metric spaces by using the continuous map φ and a nonnegative constant δ.

Keywords: (α,β)-orbital-cyclic functions, Fixed point, Extended cone metric space, Cone metric space

Introduction and Preliminaries

Cone metric spaces are a generalization of traditional metric spaces that allow for more flexible notions of distance and convergence. In a cone metric space, the usual notion of distance is replaced by a cone metric, which measures the distance between points in a way that is not constrained by the triangle inequality as in traditional metric spaces. Cone metric spaces are used in various mathematical contexts, including functional analysis and topology. These spaces are used in various fields, including mathematics, computer science, and engineering, to model situations where distances between points are not well-defined or are subject to certain uncertainties. Cone metric spaces are a way to extend the concept of a metric space to a more flexible and versatile setting. Let us start with the cone definition.

Definition 1.1 Das and Bag (2022). Let P be a nonempty subset of E, where E is an ordered Banach space. The set P is called a *cone* if and only if:

(i) P is closed, nonempty and $P \neq \{\theta\}$,

(ii) $a,b \in \mathbb{R}$; $a,b \geq 0$, $x,y \in P$ implies $ax+by \in P$,

(iii) $x \in P$ and $-x \in P$ implies $x=\theta$.

The cone P is called *normal* if there is a positive real number K such that, for all x, y in P we have:

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leq K\|y\| \tag{1}$$

or, equivalently, for three given sequences with terms from P, the sandwich rule holds i.e.

$$x_n \preceq y_n \preceq z_n \text{ and } \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = c \text{ implies that } \lim_{n\to\infty} y_n = c \tag{2}$$

The positive number K is called *the normality constant* of P.

The cone P is called *regular* if every increased sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n=1}^{\infty}$ is a sequence such $x_1 \leq x_2 \leq \cdots \leq b$ for some $b \in E,$ then there is a $x \in E,$ such that $\lim_{n\to\infty}\|x_n - x\| = 0.$ Equivalently, the cone P is called *regular* if every decreasing sequence, which is bounded from below is convergent. Regular cones are normal and there exist normal cones which are not regular.

In a cone $P \subset E,$ a partial ordering relation $\preceq$ with respect to the cone P is defined as follows:

$$x \preceq_P y \text{ iff } -x \in P \text{ for all } x, y \in P. \tag{3}$$

We also write $x \prec y$ if, $x \preceq y$ and $x \neq y,$ while for every $x, y \in P,$ $x \preceq_P y$ if f $y - x \in Int(P).$ In the following, always P is a cone in the ordered Banach space E.

Definition 1.2 Das and Bag (2022). Let P be a cone and X a non-empty set. The function $d: X \times X \to P$ is called a *cone metric* if it satisfies the following conditions:

($c1$) $d(x,y) \succ \theta$ for all $x, y \in X,$ and $d(x,y) = \theta$ iff $x = y,$
($c2$) $d(x,y) = d(y,x)$ for all $x, y \in X,$
($c3$) $d(x,z) \preceq d(x,y) + d(y,z)$ for all $x, y, z \in X.$

Then d is called *cone* metric on X and the pair (X, d) is called a *cone* metric space. Czerwik and Bakhtin introduced the concept of b-metric space and generalized the Banach contraction principle in such spaces.

Definition 1.3 Czerwik (1993). Let be X a nonempty set and $s \geq 1$ be a given real number. A function $d: X \times X \to \mathbb{R}^+$ is called a b-metric if, for all $x, y, z \in X$ it satisfies the conditions:

($b1$) $d(x,y) = 0$ iff $x = y,$
($b2$) $d(x,y) = d(y,x),$
($b3$) $d(x,z) \leq s[d(x,y) + d(y,z)].$

The pair (X, d) is called b-metric space with parameter s.
According to this definition, the cone b-metric notion will be defined as follows:

Definition1.4 Czerwik (1993). Let be X a nonempty set and $s \geq 1$ be a given real number. Suppose the mapping $d: X \times X \to E$ satisfies the conditions:

(i) $d(x,y) \succ \theta$ for all $x, y \in X$ and $d(x,y) = \theta$ iff $x = y,$
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X,$
(iii) $d(x,z) \preceq s[d(x,y) + d(y,z)]$ for all $x, y, z \in X.$

Then d is called a cone b-metric on X and the pair (X, d) is called a cone b-metric space.

As a development of the b-metric spaces, Kamran et al. [2] in 2017, introduced the notion of extended *b*-metric spaces taking instead of the coefficient *s*, a two-variable function θ.

Definition 1.5 Das and Bag (2022). Let X be a nonempty set and $\theta : X \times X \to [1,+\infty)$. A function $d_\theta : X \times X \to [0,+\infty)$ is an *extended cone metric,* if for all $x, y, z \in X$ it satisfies:

$(d_\theta 1)$ $d_\theta(x,y) = 0$ iff $x = y$;
$(d_\theta 2)$ $d_\theta(x,y) = d_\theta(y,x)$, for all $x, y \in X$;
$(d_\theta 3)$ $d_\theta(x,z) \le \theta(x,z)\left[d_\theta(x,y) + d_\theta(y,z)\right]$ for all $x, y, z \in X$.
The pair $\left(X, d_\theta\right)$ is called an *extended cone metric space.*

Remark 1.6 Das and Bag (2022). If $\theta : X \times X \to [1,+\infty)$, where $\theta(x,y) = s$, for $x, y \in X$, where $s \in [1,+\infty)$ is a constant, then we obtain the *b*-metric space definition, and when *s*=1 we obtain the cone metric space definition.

Example 1.7 Czerwik (1993). Let $X = \{1, 2, 3, \ldots\}$. Define $\theta : X \times X \to [1,+\infty)$ and $d_\theta : X \times X \to \mathbb{R}^+$ as:

$$\theta(x,y) = \begin{cases} |x-y| & \text{for } x \ne y \\ 1 & \text{for } x = y \end{cases} \text{ and } d_\theta(x,y) = (x-y)^4$$

The pair (X, d_θ) is an extended *b*-metric space.

In 2022, Das and Bag extended the function θ from $X \times X$ to $X \times X \times X$ obtaining this

Definition 1.8 Das and Bag (2022). Let be X a nonempty set and $\theta : X \times X \times X \to [1+\infty)$. Let $d_\theta : X \times X \to \mathbb{R}^+$ be a function which satisfies the following conditions:

$(d_\theta 1)$ $d_\theta(x,y) \ge 0$ for all x, y and $d_\theta(x,y) = 0$ iff $x = y$,
$(d_\theta 2)$ $d_\theta(x,y) = d_\theta(y,x)$ for all x, y in X,
$(d_\theta 3)$ $d_\theta\left(x,y\right) \le \theta(x,y,z)\left[d_\theta(x,z) + d_\theta(z,y)\right]$.

The function d_θ is called *extended cone metric* on X and the pair (X, d_θ) is called *extended cone metric space.*

Example 1.9 Let $X = \{2, 4, 6\}$ and $\theta : X \times X \to [1,+\infty)$ defined by $\theta(x,y) = 1 + 2x + 3y$. Let be the cone $P = \{(a,b) \in \mathbb{R}^2 : a, b \ge 0\}$. Define the function d_θ by :

$d_\theta(2,4) = d_\theta(4,2) = (10,10)$
$d_\theta(2,6) = d_\theta(6,2) = (20,20)$
$d_\theta(4,6) = d_\theta(6,4) = (30,30)$
$d_\theta(2,2) = d_\theta(4,4) = d_\theta(6,6) = (0,0) = \theta$

It is evident by the definition of d_θ that the first and second conditions of extended cone metric space are fulfilled. Let now check the third condition:

$$d_\theta(x,z) \le \theta(x,z)\left[d_\theta(x,y) + d_\theta(y,z)\right] \text{ for all } x, y, z \in X \text{ or,}$$
$$\theta(x,z)\left[d_\theta(x,y) + d_\theta(y,z)\right] - d_\theta(x,z) \ge \theta$$

$\theta(2,6)\left[d_\theta(2,4) + d_\theta(4,6)\right] - d_\theta(2,6) = 23\left[(10,10) + (30,30)\right] - (20,20) = (900,900) \in P$
$\theta(2,4)\left[d_\theta(2,6) + d_\theta(6,4)\right] - d_\theta(2,4) = 17\left[(20,20) + (30,30)\right] - (10,10) = (840,840) \in P$
$\theta(4,6)\left[d_\theta(4,2) + d_\theta(2,6)\right] - d_\theta(4,6) = 27\left[(10,10) + (20,20)\right] - (30,30) = (780,780) \in P$

Then (X, d_θ) is an extended *cone* metric space.
Some basic concepts, like convergence, Cauchy sequence, continuity and completeness in cone metric spaces are defined as follows.

Definition 1.10 Kamran et al. (2017). Consider a sequence $\{x_n\}$ in a cone metric space (X, d) and P be a normal cone in E with normal constant M. Then

1. $\{x_n\}$ converges to x if for every $c \in E$ with $c \succcurlyeq \theta$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) \preccurlyeq c$. Denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow[n\to\infty]{} x$.
2. $\{x_n\}$ is said to be Cauchy in X if for every $c \in E$ with $c \succcurlyeq \theta$, $\exists N \in \mathbb{N}$ such that for all $n, m \geq N \Rightarrow d(x_n, x_m) \preccurlyeq c$.
3. The mapping $T : X \to X$ is said to be continuous at a point $x \in X$ if for every sequence $\{x_n\}$ converging to x it follows that $\lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tx$.
4. (X, d) is said to be a complete cone metric space if every Cauchy sequence in X is convergent in X.

2 (S,T)-Orbital-Cyclic Admissible Pair

Popescu in his paper in 2014 introduced the concept of α-orbital admissible functions as follows:

Definition 2.1 Popescu (2014). Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. We say that T is an α –orbital admissible if for all $x, y \in X$, we have

$$\alpha(x,Tx) \geq 1 \quad \Rightarrow \quad \alpha(Tx,T^2x) \geq 1 \tag{4}$$

Definition 2.2. Khamsi, (2010).A set X is regular with respect to mapping α: $X \times X \to [0, \infty)$ if, whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ and $\alpha(x, x_{n(k)}) \geq 1$ for all n.
Alqahtani et al. (2018) introduced the concept of (α, β)-orbital-cyclic admissible pair and proved some fixed point results for a couple of orbital cyclic functions in extended b-metric spaces.

Definition 2.3 Alqahtani et al. (2018). Let S, T are two self-mappings on a complete extended cone metric space (X, d_θ). Suppose that there are two functions $\alpha, \beta : X \times X \to [0,\infty)$ such that for any $x \in X$,

$$\begin{aligned} &\alpha(x,Tx) \geq 1 \quad \Rightarrow \quad \beta(Tx,STx) \geq 1 \quad \text{and} \\ &\beta(x,Sx) \geq 1 \quad \Rightarrow \quad \alpha(Sx,TSx) \geq 1 \end{aligned} \tag{5}$$

Then the pair (S, T) is called (α, β)*-orbital -cyclic admissible* pair.

Main Result

Below, we give this key-lemma which is essential for our main results.

Lemma 2.4 *Let (X, d_θ) be an extended cone metric space and let P be a normal cone with normality constant K. If there exists $q \in [0,1)$ such that the sequence $\{x_n\}$ for an arbitrary $x_0 \in X$ satisfies*

$$\lim_{n,m\to\infty} \theta(x_n, x_m, x_{n+1}) < \frac{1}{q} \quad \textit{and also}$$

$$\theta \prec d_\theta(x_n, x_{n+1}) \circ \; q d_\theta(x_{n-1}, x_n) \quad \textit{for any } n \in \mathbb{N}, \tag{6}$$

then the sequence $\{x_n\}$ is Cauchy in X.

Proof. Using the inequality (6) recursively it follows that

$$\theta \circ \; d_\theta(x_n, x_{n+1}) \circ \; q^n d_\theta(x_1, x_0) \tag{7}$$

Taking the limits in both sides for *n* to infinity and using the fact that $q < 1$, it follows that

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = 0. \tag{8}$$

From $(d_\theta 3)$ property of extended cone metric and the triangular inequality we can write as follows

$$\begin{aligned} d_\theta(x_{n+p}, x_n) \circ \; & \theta\left(x_{n+p}, x_{n+p-1}, x_n\right)\cdot\left(d_\theta(x_{n+p}, x_{n+p-1}) + d_\theta(x_{n+p-1}, x_n)\right) \\ & \circ \; \theta\left(x_{n+p}, x_{n+p-1}, x_n\right)\cdot\theta\left(x_{n+p-1}, x_{n+p-2}, x_n\right)\cdot\left(d_\theta(x_{n+p-1}, x_{n+p-2}) + d_\theta(x_{n+p-2}, x_n)\right) \\ & +\theta\left(x_{n+p}, x_{n+p-1}, x_n\right)\cdot d_\theta(x_{n+p}, x_{n+p-1}) \\ & \circ \; \theta\left(x_{n+p}, x_{n+p-1}, x_n\right)\cdot\theta\left(x_{n+p-1}, x_{n+p-2}, x_n\right)\cdots\theta\left(x_{n+2}, x_{n+1}, x_n\right)\cdot q^n\cdot d_\theta(x_1, x_0) \\ & +\cdots+\theta\left(x_{n+p}, x_{n+p-1}, x_n\right)\cdot q^{n+p-1}\cdot d_\theta(x_1, x_0) \end{aligned} \tag{9}$$

Taking the norm for both sides of the final inequality, it follows that:

$$\left\|d_\theta(x_{n+p}, x_n)\right\| \le K\left[\theta\left(x_{n+p}, x_{n+p-1}, x_n\right)\cdot\theta\left(x_{n+p-1}, x_{n+p-2}, x_n\right)+\cdots+\theta\left(x_{n+2}, x_{n+1}, x_n\right)\cdot q^{n+p-1}\right]\left\|d_\theta(x_1, x_0)\right\| \tag{10}$$

Remember that *K* is the normality constant of the cone *P*.

Consequently,

$$\left\|d_\theta(x_{n+p}, x_n)\right\| \le K\cdot\left(S_{n+p} - S_n\right)\cdot\left\|d_\theta(x_1, x_0)\right\| \tag{11}$$

where S_n are the partial sums of the convergent series $\sum_{i=1}^{\infty} q^p \cdot \prod_{i=2}^{p} \theta(x_{n+i}, x_{n+i-1}, x_n)$.

Then,

$$\lim_{n,p\to\infty}\left\|d_\theta(x_{n+p}, x_n)\right\| = 0$$

which implies that the sequence $\{x_n\}$ is Cauchy.

Theorem 2.5. *Let T, S be two self-mappings on a complete extended cone metric space* (X, d_θ) *such that the pair (T, S) is* (α,β)*-orbital-cyclic admissible. Suppose that there is a constant* $\delta > 0$ *and a continuous comparison function* $\varphi: X \to E$ *such that* $\lim_{n\to\infty}\left\|\varphi^n(t)\right\| = 0$ *and*

(i) *for each* $x_0 \in X$, $\lim_{n,m\to\infty}\theta(x_n, x_m, x_{n+1}) < \frac{1-k}{k}$, *where* $x_{2n} = Sx_{2n-1}$ *and* $x_{2n+1} = Tx_{2n}$ *for each* $n \in \Box$,
(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \ge 1$,
(iii) *S and T are continuous and satisfies the following inequality*

$$\begin{aligned} & \alpha(x,Tx)\beta(y,Sy)d_\theta(Tx,Sy) \circ \; \varphi\left(\max\left\{d_\theta(x,Tx),\, d_\theta(y,Sy)\right\}\right) \\ & +\delta\min\left\{d_\theta(x,Tx), d_\theta(y,Sy), d_\theta(Tx,y), d_\theta(x,Sy)\right\}. \end{aligned} \tag{12}$$

Then the pair of mappings (T,S) *has a common fixed point u, that is* $Tu = u = Su$.

Proof. By assumption (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Let be $x_1 = Tx_0$ and $x_2 = Sx_1$. Continuing in this way inductively, we construct the sequence $\{x_n\}$ where

$$x_{2n} = Sx_{2n-1} \quad \text{and} \quad x_{2n+1} = Tx_{2n} \quad \forall n \in \mathbb{N} \tag{13}$$

We have $\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1$ and since (*S*, *T*) is an (α, β) -orbital-cyclic admissible pair, it follows that

$$\alpha(x_0, x_1) \geq 1 \quad \Rightarrow \quad \beta(x_0, STx_0) = \beta(x_1, x_2) \geq 1$$

and

$$\beta(x_1, x_2) \geq 1 \quad \Rightarrow \quad \alpha(Sx_1, TSx_1) = \alpha(x_2, x_3) \geq 1$$

Applying again (5),

$$\alpha(x_2, x_3) \geq 1 \quad \Rightarrow \quad \beta(Tx_2, STx_2) = \beta(x_3, x_4) \geq 1$$

and

$$\beta(x_3, x_4) \geq 1 \quad \Rightarrow \quad \alpha(Sx_3, TSx_3) = \alpha(x_4, x_5) \geq 1.$$

Recursively, we obtain

$$\alpha(x_{2n}, x_{2n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N} \tag{14}$$

and

$$\beta(x_{2n+1}, x_{2n+2}) \geq 1 \quad \text{for all } n \in \mathbb{N} \tag{15}$$

We can continue the proof assuming that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. This assumption doesn't loss the generality of the proof. Indeed, if there exists any $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, the common fixed point of mappings *S*, *T* is $u = x_{n_0}$. To see that $u = x_{n_0}$ is the requested point, we study the following two cases.

(i) If $n_0 = 2k$, we have $x_{2k} = x_{2k+1} = Tx_{2k}$. Thus, x_{2k} is a fixed point of *T*. To show that x_{2k+1} is also a fixed point of *S* we must prove that $x_{2k+1} = Sx_{2k+1}$ or, equivalently $Tx_{2k} = Sx_{2k+1}$.

Suppose on the contrary, that $d_\theta(x_{2k+1}, Sx_{2k+1}) \succ \theta$. Replacing $x = x_{2k}$ and $y = x_{2k+1}$ in (12) and using (14), (15) we obtain:

$$\begin{aligned}
\theta \prec\ & d_\theta(x_{2k+1}, x_{2k+2}) = d_\theta(Tx_{2k}, Sx_{2k+1}) \\
\circ\ & \alpha(x_{2k}, Tx_{2k})\beta(x_{2k+1}, Sx_{2k+1})d_\theta(Tx_{2k}, Sx_{2k+1}) \\
\circ\ & \varphi\left(\max\{d_\theta(x_{2k}, Tx_{2k}), d_\theta(x_{2k+1}, Sx_{2k+1}),\}\right) + \delta \min\left\{d_\theta(x_{2k}, x_{2k+1}), d_\theta(x_{2k}, Tx_{2k}), d_\theta(x_{2k+1}, Tx_{2k+1})\right\}
\end{aligned}$$

By replacing $d_\theta(x_{2k}, Tx_{2k}) = \theta$, $d_\theta(x_{2k}, x_{2k+1}) = \theta$ and using the fact that $\varphi(t) < t$ we obtain

$$\theta \prec d_\theta(x_{2k+1}, x_{2k+2}) \circ \varphi(d_\theta(x_{2k+1}, x_{2k+2})) \prec d_\theta(x_{2k+1}, x_{2k+2}) \tag{16}$$

which is a contradiction. Thus, $d_\theta(x_{2k+1}, Sx_{2k+1}) = \theta$ or $d_\theta(Tx_{2k}, Sx_{2k+1}) = 0$ and $x_{2k} = x_{2k+1} = Tx_{2k} = Sx_{2k+1}$ which implies that $u = x_{2k} = x_{2k+1}$ is a common fixed point of *T* and *S*.

(ii) If $n_0 = 2k+1$ we can proceed analogously obtaining the same result.

From now on, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. For this, it is enough to prove that our sequence fulfills the requirements of Lemma 2.4.

$$\begin{aligned}
& \theta \prec d_\theta(x_{2n+1}, x_{2n+2}) \leq \alpha(x_{2n,}, Tx_{2n})\beta(x_{2n+1}, Sx_{2n+1})d_\theta(Tx_{2n}, Sx_{2n+1}) \\
& \circ\ \varphi\Big(\max\left\{d_\theta(x_{2n,}Tx_{2n}), d_\theta(x_{2n+1}, Sx_{2n+1})\right\}\Big) \\
& +\delta \min\left\{d_\theta(x_{2n,}Tx_{2n}), d_\theta(x_{2n+1}, Sx_{2n+1}), d_\theta(Tx_{2n}, x_{2n+1}), d_\theta(x_{2n}, Sx_{2n+1})\right\}
\end{aligned} \tag{17}$$

Since $d_\theta(Tx_{2n}, x_{2n+1}) = d_\theta(x_{2n+1}, x_{2n+1}) = \theta$, it follows that

$$\min\left\{d_\theta(x_{2n,}Tx_{2n}), d_\theta(x_{2n+1}, Sx_{2n+1}), d_\theta(Tx_{2n}, x_{2n+1}), d_\theta(x_{2n}, Sx_{2n+1})\right\} = \theta$$

Therefore, the inequality (17) will be write more simply

$$\begin{aligned} \theta \prec_\theta\ & (x_{2n+1}, x_{2n+2}) \circ\ \alpha(x_{2n,}Tx_{2n})\beta(x_{2n+1}, Sx_{2n+1})d_\theta(Tx_{2n}, Sx_{2n+1}) \\ \circ\ & \varphi\Big(\max\left\{d_\theta(x_{2n,}Tx_{2n}), d_\theta(x_{2n+1}, Sx_{2n+1})\right\}\Big) \end{aligned} \tag{18}$$

Further, using the property of maximal element of a set and the monotony of φ, we have

$$\begin{aligned} d_\theta(x_{2n+1}, x_{2n+2}) \circ\ & \varphi\Big(\max\left\{d_\theta(x_{2n,}Tx_{2n}),\ d_\theta(x_{2n+1}, Sx_{2n+1})\right\}\Big) \\ & = \varphi\Big(\max\left\{d_\theta(x_{2n,}x_{2n+1}),\ d_\theta(x_{2n+1}, x_{2n+2})\right\}\Big) \end{aligned} \tag{19}$$

If $\max\left\{d_\theta(x_{2n,}x_{2n+1}),\ d_\theta(x_{2n+1}, x_{2n+2})\right\} = d_\theta(x_{2n+1}, x_{2n+2})$ then from the inequality $\varphi(t) < t$ it follows that

$$d_\theta(x_{2n+1}, x_{2n+2}) \prec d_\theta(x_{2n+1}, x_{2n+2})$$

which is contradiction. As a consequence, we have

$$\max\left\{d_\theta(x_{2n,}x_{2n+1}),\ d_\theta(x_{2n+1}, x_{2n+2})\right\} = d_\theta(x_{2n}, x_{2n+1}).$$

Therefore, from (19)

$$d_\theta(x_{2n+1}, x_{2n+2}) < \varphi\Big(d_\theta(x_{2n,}x_{2n+1})\Big) \tag{20}$$

From Lemma 2.4 and the first condition of our theorem, it follows that $\{x_n\}$ is a Cauchy sequence in *X*. Since (X, d_θ) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{21}$$

From (21) it is clear that as subsequences of $\{x_n\}$, we have that $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converges at the point $u \in X$

$$\lim_{n\to\infty} x_{2n} = \lim_{x\to\infty} x_{2n+1} = u. \tag{22}$$

By using the continuity of S and T we obtain

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\lim_{n\to\infty} x_n = Tu$$

and

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Sx_n = S\lim_{n\to\infty} x_n = Su.$$

Thus, the point *u* is a common fixed point for the pair of mappings (S, T).
Now, before we prove the uniqueness of *u*, we accept that for all *x*, *y* common fixed point of mappings *S* and *T* (denoted $x, y \in CT(S,T)$) we have

$$\alpha(x, Tx) \geq 1 \text{ and } \beta(y, Sy) \geq 1. \tag{23}$$

Suppose on the contrary that *v* is another fixed point of the mapping T where $u \neq v$. Therefore, $u = Tu = Su$ and $v = Tv = Sv$. From (23) we derive that

$$\alpha(u,Tu)\geq 1 \text{ and } \beta(v,Sv)\geq 1. \tag{24}$$

Using the inequality (12), for *T* we have

$$\alpha(u,Tu)\beta(v,Sv)d_\theta(Tu,Sv) \circ \ \varphi\left(\max\left\{d_\theta(u,Tu),\, d_\theta(v,Sv)\right\}\right)$$
$$+\delta\min\left\{d_\theta(u,Tu), d_\theta(v,Sv), d_\theta(Tu,v), d_\theta(v,Su)\right\}.$$

Since $u,v\in CT(S,T)$), from the last inequality, we get that

$$\begin{aligned} d_\theta(Tu,Sv) &\circ \ \varphi\left(\max\left\{\theta,\theta\right\}\right)+\delta\min\left\{\theta,\theta,d_\theta(Tu,v),d_\theta(v,Su)\right\} \\ &=\varphi(\theta)+\delta\cdot\theta \\ &=\theta. \end{aligned}$$

Finally, the inequality $d_\theta(Tu,Sv)=d_\theta(u,v)\leq\theta$ is possible only when $d_\theta(u,v)=\theta$ or $u=v$ which is a contradiction. Thus, *u* is unique fixed point of *T*.

2.1 (α,β) -Orbital-Cyclic Admissible Mappings

Definition 2.1.1 Alqahtani et al. (2018). Let *X* be a nonempty set, $T:X\to X$ $\alpha,\beta:X\times X\to[0,\infty)$. We say that *T* is an (α,β) -orbital-cyclic admissible mapping if

$$\begin{aligned} &\alpha(x,Tx)\geq 1 \text{ implies } \beta(Tx,T^2x)\geq 1 \qquad \text{and} \\ &\beta(x,Tx)\geq 1 \text{ implies } \alpha(Tx,T^2x)\geq 1 \end{aligned} \tag{25}$$

Corollary 2.1.2 *Let T be a self-mapping on a complete extended cone metric space* (X,d_θ) *such that the mapping T forms an* (α,β) *-orbital-cyclic admissible mapping. Suppose that there is a constant* $\delta>0$ *and a continuous comparison function* $\varphi:X\to E$ *such that* $\lim_{n\to\infty}\left\|\varphi^n(t)\right\|=0$ *and*

(i) *for each* $x_0\in X$, $\lim_{n,m\to\infty}\theta(x_n,x_m,x_{n+1})<\dfrac{1-k}{k}$, *where* $x_{2n}=Sx_{2n-1}$ *and* $x_{2n+1}=Tx_{2n}$ *for each* $n\in\square$,
(ii) *there exists* $x_0\in X$ *such that* $\alpha(x_0,Tx_0)\geq 1$,
(iii) *T is continuous and satisfies this inequality*

$$\begin{aligned} &\alpha(x,Tx)\beta(y,Ty)d_\theta(Tx,Ty)\leq\varphi\left(\max\left\{d_\theta(x,Tx),\, d_\theta(y,Ty)\right\}\right) \\ &+\delta\min\left\{d_\theta(x,Tx), d_\theta(y,Ty), d_\theta(Tx,y), d_\theta(x,Ty)\right\} \end{aligned} \tag{26}$$

then T has a fixed point $u=Tu$.
Proof. If we replace $S=T$ in the Theorem 2.5 the result is evident.

Corollary 2.1.3 *Let T be a self-mapping on a complete extended cone metric space* (X,d_θ) *such that the mapping T forms an* (α,β) *-orbital-cyclic admissible mapping. Suppose that there is a constant* $\delta>0$ *and a continuous comparison function* $\varphi:X\to E$ *such that* $\lim_{n\to\infty}\left\|\varphi^n(t)\right\|=0$ *and*

(i) *for each* $x_0\in X$, $\lim_{n,m\to\infty}\theta(x_n,x_m,x_{n+1})<\dfrac{1-k}{k}$, *where* $x_{2n}=Sx_{2n-1}$ *and* $x_{2n+1}=Tx_{2n}$ *for each* $n\in\square$,
(ii) *there exists* $x_0\in X$ *such that* $\alpha(x_0,Tx_0)\geq 1$,
(iii) *T is continuous and satisfies this inequality*

$$\begin{aligned}&\alpha(x,Tx)\alpha(y,Ty)d_\theta(Tx,Ty)\le\varphi\left(\max\left\{d_\theta(x,Tx),\, d_\theta(y,Ty)\right\}\right)\\&+\delta\min\left\{d_\theta(x,Tx),d_\theta(y,Ty),d_\theta(Tx,y),d_\theta(x,Ty)\right\}\end{aligned} \tag{27}$$

then T has a fixed point $u=Tu$.
Proof: By taking $\beta(x,y)=\alpha(x,y)$ in Corollary 2.1.2, the proof is done.

Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

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Author Information

Zamir Selko
University of Elbasan “Aleksandër Xhuvani”
Elbasan, Albania
Contact e-mail: zamir.selko@uniel.edu.al

Eriola Sila
University of Tirana,
Tirana, Albania

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