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On Fixed Point Results for Nonlinear Contractions in Fuzzy Cone Metric Space

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Abstract: The study of Fixed Point Theory in various metric space has been on focus of scientific development for many authors. It has been advanced either by generalizing the contractive inequality or by extending the conditions of metric. Fuzzy metric space has been defined as space in which the distance between elements is not an exact number in difference with metric space. Fixed point Theory is an important framework point of view in fuzzy metric spaces. Many studies have been showed the existence and uniqueness of a fixed point for different type of contractions in these spaces. Nonlinear contractions and their generalizations have been under investigations in several metric spaces. The aim of this paper is the study of fixed points for generalized nonlinear contractions in fuzzy metric space. Our results guarantee the existence and uniqueness of a fixed point for these contractions and extend some known theorems in metric space to fuzzy metric space. As an application of main theorem an example is taken.

Keywords: Fixed point, Fuzzy cone metric space, Generalized nonlinear contraction, Auxiliary function

1. Introduction

The Fuzzy metric space was introduced by Kramosil and Michalek (1975) as an approach of probabilistic space in Fuzzy sets Theory studied by Zadeh (1965). In 1994, George and Veeramani defined GV Fuzzy metric space by replacing the last condition of Fuzzy metric with a stronger one. The concept of cone metric space was introduced by Huang and Zhang (2007) by generalizing metric using a subset of an ordered Banach space.

In 2015, Öner et al. generalized GV Fuzzy metric space to Fuzzy cone metric space. They studied the topology of this space, its metrizability (Oner et al., 2016) and showed some fixed-point results in them (Oner et al., 2016). There are many authors, who have given their contribution in studying Fixed point theory on Fuzzy cone metric spaces. (Rehman & Li, 2017; Chen et al., 2020) .In this paper, we establish and prove some fixed point results on fuzzy cone metric space related to generalized nonlinear contractions.

2. Preliminaries

Firstly, we begin with some main concepts on Fuzzy metric.

Definition 2.1 Schweizer and Sklar (1960) A continuous t-norm is called an operation $*: [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions:

1. $*$ is associative, commutative and continuous;
2. For each $u_{0,}, u_1, v_0, v_1 \ \in [0,1] u_0 \leq u_1 and\ v_0 \leq v_1$ it yields $1 * u_0 = v_0,\ u_0 * v_0 \leq u_1 * v_1$

Examples (Schweizer and Sklar 1960)

The operations defined as below:

$*: [0,1] \times [0,1] \to [0,1]$

1. $U * V = UV$ (t-norm product)
2. $U * V = max\{\mathrm{U} + \mathrm{V}, -1, 0\}$ $\to$ Lukasiericz t-norm)

are continuous t-norms.

Definition 2.2 (Kramosil & Michalek ,1975) Let X be an arbitrary set, $*$ is a continuous t-norm, M is a Fuzzy set on $X^2 \times (0, \infty)$. The 3 – tuple (X, M, $*$) is called a Fuzzy metric space if it completes the conditions below:

(FM_1) $For\ each\ x, y \in X,\ t \in (0, \infty)\ M(x, y, t) > 0$;
(FM_2) $For\ each\ x, y \in X^2 \times (0, \infty), M(x, y, t) = M(y, x, t)$;
(FM_3) $M(x, y, t)\ * M(y, z, s) \leq M(x, z, t + s)$ for each $x, y \in X^2$ and $t, s \in (0, \infty)$;
(FM_4) $M(x, y, \cdot): (0, \infty) \to [0,1]$ is left continuous.

In 1994, George and Veeramani generalized the concept of Fuzzy metric space to GV Fuzzy metric space as follows.

Definition 2.3 Let X be an arbitrary set, , $*$ is a continuous t-norm, M is a Fuzzy set on $X^2 \times (0, \infty)$.
Is a continuous t-norm, and M a Fuzzy set on $X^2 \times [0, \infty)$.

$(X, M, *)$ is called GV-fuzzy metric space if it satisfies the following conditions:
($GVFM_1$) For each $(x, y, t) \in X^2 \times [0, \infty), M(x, y, t) > 0$;
($GVFM_2$) For each $(x, y, t) \in X^2 \times [0, \infty),\ M(x, y, t) = 1$ iff $x = y$;
($GVFM_3$) For each $(x, y, t) \in X^2 \times [0, \infty),\ M(x, y\ , t) = M(y, x, t)$;
($GVFM_4$) For each $(x, y, t, s) \in X^2 \times [0, \infty) \times [0, \infty),\ M(x, y\ , t + s) \geq M(x, y, t) * M(x, y, s)$.
($GVFM_5$) For each $(x, y) \in X^2$ the function $M(x, y, \cdot): (0, \infty) \to\ [0,1]$ is continuous.

Below, there is given the concept of cone.

Let E be a real Banach space.

Definition 2.4 (Huang & Zhang, 2007) The subset $P \subseteq E$ that satisfies the following conditions:

1. P is closed, P$\neq \Phi$ and P$\neq \{0\}$;
2. For each $\alpha, \beta \in [0, \infty)$ and x, y$\in P, then\ \alpha x + \beta y \in P$;
3. For each x$\in P, if\ (-x) \in P\ then\ x = 0$.

Is called a cone.
In the cone P a partial ordering "≤" is defined as below: For every $x, y \in P\ x \leq y$ iff $x - y \in int\ P$.

Suppose that $int\ P \neq \Phi$.

Definition 2.5 (Öner et al., 2015)

Let $P \subseteq E$ be a cone with nonempty interior, X be an arbitrary set, $*$ a continuous t-norm and M a fuzzy set on $X \times X \times int\ P$. The 3-tuple $(X, M, *)$ is called a Fuzzy cone metric space if it satisfies the following conditions:

(FCM_1): For each $(x, y)\ \in X \times X$, $t \in int\ P$, $M(x, y, t) > 0$ and $M(x, y, t) = 1$ iff $x = y$;

(FCM_2): For each $(x, y, t) \in X \times X \times int\ P$, $M(x, y, t) = M(y, x, t)$;
(FCM_3): For each $x, y, z \in X$ and $t, s \in Int\ P$, $M(x, y, t) * M(y, z, x) \leq M(x, z, t + s)$;
(FCM_4): $M(x, y, \cdot): int\ P \to [0, 1]$is continuous.

Below, there are given the definition of convergent sequences, Cauchy sequences (Öner et al. 2015) in a Fuzzy cone metric space.

Let $(X, M, *)$ be a Fuzzy cone metric space and $(xn)\ n \in N$ a sequence in X and $x \in X$.
The sequence $(x_n)_{n \in N}$ is called convergent to x if for each $r \in (0, 1)$ there exist a natural number $n_0 \in N$, such that for every natural number $n > n_0$, $M(x_n, x, t) > 1 - r$ for $t > 0$.
The sequence $(x_n)_{n \in N}$ is called Cauchy if for each $r \in (0, 1)$, there exist $n_0 \in N$
that for each $n, m > n_0$, $M(x_n, x_m, t) > 1 - r$.
A Fuzzy cone metric space is called complete if every Cauchy sequence is convergent on it.

Definition 2.6 (Alam et al., 2022)

Let $\varphi: [0, \infty) \to [0, \infty)\ and\ \psi: [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$

1. $\varphi\ is\ right\ continuous$;
2. $\varphi\ is\ monoton\ increasing$;
3. $\psi(t) > 0$, $t > 0$
4. $\lim_{t \to r} inf\ \psi(t) > 0, for\ every\ r > 0$.

The function $T: X \to X$ that satisfies the inequality

$$\varphi(d(Tx, Ty) \leq \varphi(d(x, y)) - \psi(d(x, y))$$

is called generalized nonlinear contraction in metric space (X, d).

Initiating from above we extend this concept to fuzzy metric spaces
The function $T: X \to X$ that satisfies the inequality:

$$\varphi\left(\frac{1}{M(Tx, Ty, t)} - 1\right) \leq \varphi\left(\frac{1}{M(x, y, t)} - 1\right) - \psi\left(\frac{1}{M(x, y, t)} - 1\right)$$

For every $x, y \in X$ is called generalized nonlinear contraction on Fuzzy cone metric space $(X, M, *)$

3. Main Results

In this section, we prove some fixed point results related to generalized nonlinear contractions on Fuzzy cone metric space.

Theorem 3.1 Let$(X, M, *)$ be a complete Fuzzy metric space and $T: X \to X$ a continuous generalized nonlinear contraction. Then it has a unique fixed point on X.

Proof. Let x_0 be an arbitrary element of X and $\{x_n\}_{n \in N}$ is the iterative sequence constructed as follows:
$x_1 = Tx_0, x_2 = Tx_1, \dots, x_n = Tx_{n-1}, \dots$.

If there exist any $x_k \in X$ such that $x_k = x_{k+1}$, then $Tx_k = x_k$ and x_k is the fixed point of function T.

Suppose that for each $n \in N, x_{n+1} \neq x_n$, and see

$$\varphi\left(\frac{1}{M(x_{n+1}, x_n, t)} - 1\right) \leq \varphi\left(\frac{1}{M(x_n, x_{n-1}, t)} - 1\right) - \psi\left(\frac{1}{M(x_n, x_{n-1}, t)} - 1\right) \leq \varphi\left(\frac{1}{M(x_n, x_{n-1}, t)} - 1\right).$$

As a result, we have that $\frac{1}{M(x_{n+1}, x_n, t)} - 1 \leq \frac{1}{M(x_n, x_{n-1}, t)} - 1$, for each $n \in N$, and the sequence $\left\{\frac{1}{M(x_n, x_{n-1}, t)} - 1\right\}_{n \in N}$ is monotone decreasing.

This sequence is bounded from below by zero $\frac{1}{M(x_n,x_{n-1},t)} - 1 > 0$.

As a consequence, it is convergent to a point $m \geq 0$

$$\lim_{n\to\infty}\left(\frac{1}{M(x_n,x_{n-1},t)} - 1\right) = m \geq 0$$

Let us prove that $m = 0$.

Suppose that $m > 0$. Since φ is continuous we have $\lim_{n\to\infty} sup\ \varphi\left(\frac{1}{M(x_n,x_{n-1},t)} - 1\right) = \varphi(m)$
Using the following inequality

$$\lim_{n\to\infty}\varphi\left(\frac{1}{M(x_{n+1},x_n,t)} - 1\right) \leq \varphi\left(\frac{1}{M(x_n,x_{n-1},t)} - 1\right) - \psi\left(\frac{1}{M(x_n,x_{n-1},t)} - 1\right)$$

we take

$$\lim_{n\to\infty} sup\ \varphi\left(\frac{1}{M(x_{n+1},x_n,t)} - 1\right) \leq \lim_{n\to\infty} sup\ \varphi\left(\frac{1}{M(x_n,x_{n-1},t)} - 1\right) + \lim_{n\to\infty} sup\ -\psi\left(\frac{1}{M(x_n,x_{n-1},t)} - 1\right) =$$

$$= \lim_{n\to\infty} sup\ \varphi\left(\frac{1}{M(x_n,x_{n-1},t)} - 1\right) - \lim_{n\to\infty} inf\ \psi\left(\frac{1}{M(x_n,x_{n-1},t)} - 1\right).$$

Consequently, the following relationship holds:

$\varphi(m) \leq \varphi(m) - \psi(m)$ and $\psi(m) \leq 0$ which means that $m = 0$

As a results, $\lim_{n\to\infty}\frac{1}{M(x_{n+1},x_n,t)} - 1 = 0$

The next step is to show that the sequence $\{x_n\}_{n\in N}$ is Cauchy.
Suppose that the sequence $\{x_n\}_{n\in N}$ is not Cauchy.
As a result there exist $r_0 > 0$ and the sequence $\{x_{nk}\}_{k\in N}, \{x_{nl}\}_{l\in N}$ such that $m_k > m_l$ and $M(x_{nk}, x_{nl}, t) < 1 - r_0 \leq M(x_{nk}, x_{nl-1}, t)$ or $\frac{1}{M(x_n,x_{n-1},t)} - 1 > \frac{1}{1-r_0} - 1 = \frac{r_0}{1-r_0} \geq \frac{1}{M(x_{nk},x_{nl},t)} - 1$.

From the inequality

$$\varphi\left(\frac{1}{M(x_{nk},x_{nl},t)} - 1\right) \leq \varphi\left(\frac{1}{M(x_{nk-1},x_{nl-1},t)} - 1\right) - \psi\left(\frac{1}{M(x_{nk-1},x_{nl-1},t)} - 1\right)$$

and using $\lim_{n\to\infty} sup\ \varphi\left(\frac{1}{M(x_{nk},x_{nl},t)} - 1\right) = \varphi\left(\frac{r_0}{1-r_0}\right)$,

we have

$$\varphi\left(\frac{r_0}{1-r_0}\right) \leq \varphi\left(\frac{r_0}{1-r_0}\right) - \lim_{n\to\infty} inf\ \psi\left(\frac{1}{M(x_{nk-1},x_{nl-1},t)} - 1\right)$$

Which yields $\lim_{n\to\infty} inf\ \psi\left(\frac{1}{M(x_{nk-1},x_{nl-1},t)} - 1\right) \leq 0$,which is contradiction.

Consequently, the sequence $\{x_n\}_{n\in N}$ is Cauchy.

Since $(X, M,*)$ is complete, it yields that the sequence $\{x_n\}_{n\in N}$ converges to a point x^*, so $\lim_{n\to\infty} x_{n+1} = x^*$
Since T, is continuous, we have

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*.$$

From the uniqueness of the limit, we have $Tx^* = x^*$.
As a result, x^* is a fixed point of function T.

The next step is to prove the uniqueness of fixed point x^* of T.
Suppose that there exist, another fixed point $y^* \in X$, $x^* \neq y^*$such that $Ty^* = y^*$
From the inequality

$$\varphi\left(\frac{1}{M(Tx^*, Ty^*, t)} - 1\right) \leq \varphi\left(\frac{1}{M(x^*, y^*, t)} - 1\right) - \psi\left(\frac{1}{M(x^*, y^*, t)} - 1\right)$$

we have that

$$\varphi\left(\frac{1}{M(x^*, y^*, t)} - 1\right) \leq \varphi\left(\frac{1}{M(x^*, y^*, t)} - 1\right) - \psi\left(\frac{1}{M(x^*, y^*, t)} - 1\right).$$

As a result, $\psi\left(\frac{1}{M(x^*, y^*, t)} - 1\right) \leq 0$ which is a contradiction, consequently $x^* = y^*$.

Corollary 3.2 Let $(X, M,*)$ be a complete Fuzzy cone metric space and T is a nonlinear contraction that satisfies the inequality

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq \frac{1}{M(x, y, t)} - 1 - \psi\left(\frac{1}{M(x, y, t)} - 1\right)$$

Then the function T has a unique fixed point x^* in X.

Proof. Taking $\varphi(t) = t$ in the inequality

$$\varphi\left(\frac{1}{M(Tx, Ty, t)} - 1\right) \leq \varphi\left(\frac{1}{M(x, y, t)} - 1\right) - \psi\left(\frac{1}{M(x, y, t)} - 1\right)$$

We are in condition of Theorem 3.1. As a result, the function T has a unique fixed point.

Corollary 3.3 Let $(X, M,*)$ be a complete Fuzzy metric space and $T: X \to X$ a function that satisfies the Banach inequality

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq k\left(\frac{1}{M(x, y, t)} - 1\right)$$

for $0 < k < 1$. Then the function T has a unique fixed point.

Proof. By replacing $\psi(t) = (1 - k)t$ in inequality of Corollary 3.2, the result is clear.

Example 3.4

Let $X = (0, \infty)$, $\quad *$ be a continuous t-norm, and $M: X^2 \times (0, \infty) \to [0,1]$, $M(x, y) = \begin{cases} 1, & x = y \\ \frac{min(x,y)}{max(x,y)} s(t) \ , & x \neq y \end{cases}$

where $s(t) = \begin{cases} t, & t \leq 1 \\ 1, & t > 1 \end{cases}$. Taking the cone $P =]0, \infty[$ with $int\ P \neq 0$, $(X, M,*)$ is a complete Fuzzy cone metric space.

Choosing $\varphi(t) = \frac{t}{2}$, $\psi(t) = \frac{2}{5}t$, $Tx = \frac{1}{10}x + \frac{1}{10}$, we see that

$$\varphi\left(\frac{1}{M(Tx, Ty, t)} - 1\right) \leq \varphi\left(\frac{1}{M(x, y, t)} - 1\right) - \psi\left(\frac{1}{M(x, y, t)} - 1\right).$$

Consequently, we are in conditions of Theorem 3.1, and the function T has a unique fixed point $x = \frac{1}{9}$

Conclusions

In this paper, there are studied the existence and uniqueness of a fixed point for generalized nonlinear contractions in Fuzzy cone metric spaces. Our results are the extensions of Jachemsky (1997) and Sutfa and Choudhury (2008) on Fuzzy cone metric spaces.

Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

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