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Curvature Analysis in a Bi-layered Non-linear Elastic Bar under Uniform Heating

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Abstract: The present theoretical paper is concerned with analysis of the curvature in a bi-layered bar of rectangular cross-section. The bar is subjected to uniform heating. The two layers of the bar have different thickness and material properties. Besides, the layers exhibit non-linear elastic mechanical behaviour that is treated by applying a power-law stress-strain constitutive relationship. The bi-layered bar curvature is studied analytically. For this purpose, the mechanical response of the bar to uniform heating is investigated. The strains in the two layers due to heating are analyzed. An equation is compiled by using the fact that the prolongations of the bar layers are equal. Two equations are worked out by considering the equilibrium of the layers. The analysis of the curvature presented in this paper is checked-up by comparing with a known solution for the curvature in a bi-layered linear-elastic bar under uniform heating. The uniform heating induced curvature in a bi-layered bar whose layers are continuously inhomogeneous in longitudinal direction is also studied.

Keywords: Curvature, Bi-layered bar, Uniform heating, Continuous inhomogeneity

Introduction

Bi-layered materials are frequently applied in various areas of up-to-date engineering. They are used for manufacturing of components of different structures, facilities and devices in car industry, aeronautics, civil engineering, power plants, optics and electronics. In principle, layered materials are preferred to traditional homogeneous engineering materials because of their superior properties especially in applications in extreme conditions (high temperature, humidity, etc.) (Dolgov, 2005; Dolgov, 2016; Kim et al., 1999). These materials are known also with their high strength-to-weight and stiffness-to-weight ratios. This fact makes them right for engineering applications where low deadweight is one of the important goals of designers (Rzhanitsyn, 1986; Kaul, 2014; Lloyd & Molina-Aldareguia, 2003; Rizov, 2017; Rizov, 2018).

Very often bi-layered components of various structures, machines and mechanisms are subjected to heating in their operation exploitation. However, the layers of a bi-layered material system usually have different coefficients of thermal expansion. This fact indicates the importance of studying various aspects of the behaviour of structural components made of bi-layered materials subjected to heating. Also, in some cases the mechanical behaviour of bi-layered material systems is non-linear elastic (this necessities application of non-linear stress-strain relationships since the Hook's law does not hold). However, the mechanical response of bi-layered structural components under heating usually is analyzed assuming linear-elastic behaviour of the material (Varvak, 1997). Besides, very often the bi-layered structural components have rectangular section.

Therefore, the basic purpose of the present paper is to analyze the curvature of a bi-layered bar of rectangular cross-section subjected to uniform heating with taking into account the material non-linearity. Two cases are studied. The first one is of a bi-layered bar with homogeneous layers. Equations for determining of axial forces and bending moment are compiled. Then axial forces are bending moment are used in the equations of equilibrium for obtaining the bar curvature induced by uniform heating. The case of a bi-layered bar with continuously inhomogeneous layers is also studied (this case is interestning mainly becouse of wide application

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of functionally graed materials (Hirai & Chen,1999; Gasik, 2010; Radhika et al., 2020, Rizov, 2019). A check-up is carried-out by comparing with a solution published in the literature.

Curvature Analysis

In this paper we deal with a bi-layered bar subjected to inform heating at temperature, Δt. The bar has a rectangular cross-section as shown in Fig. 1.

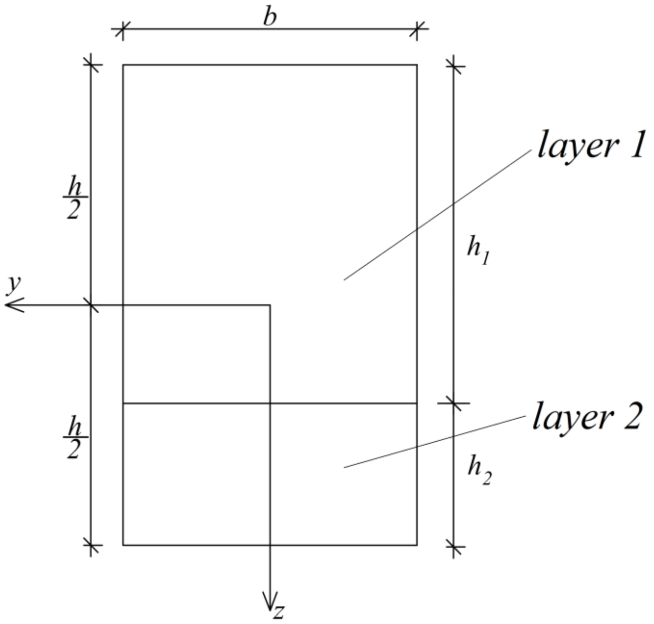

Figure 1. Cross-section of the bi-layered bar.

The width and thickness of the cross-section are denoted by h and b, respectively. The thickness of the upper layer is denoted by h_1. The lower layer thickness is h_2. The two layers have different material properties. Besides, the layers exhibit non-linear elastic mechanical behaviour. First, the case of homogeneous layers is analyzed. The mechanical behaviour of the upper layer is treated by using the following non-linear stress-strain constitutive relationship:

$$\sigma_1 = E_1 \varepsilon^{n_1}, \tag{1}$$

where σ_1 is the stress, ε is the strain, E_1 and n_1 are material properties.

The stress-strain constitutive relationship of the lower layer of the bar is written as

$$\sigma_2 = E_2 \varepsilon^{n_2}, \tag{2}$$

where σ_1, ε, E_1 and n_1 are the stress, strain and two material properties, respectively.

Our purpose is to analyze the curvature, κ, of the bi-layered bar induced by uniform heating with considering the non-linear elastic behaviour of the layers. The coefficients of thermal expansion of the upper and lower layers are denoted by α_{t1} and α_{t2}, respectively. If we assume that the layers of the bar are not connected, the prolongations, Δl_{t1} and Δl_{t2}, of the upper and lower layers can be written as

$$\Delta l_{t1} = \alpha_{t1} \Delta t\, l, \tag{3}$$

$$\Delta l_{t2} = \alpha_{t2} \Delta t\, l, \tag{4}$$

where l is the bar length. From (3) and (4), the strains, ε_1 and ε_2, in the upper and lower layers are found as

$$\varepsilon_1 = \alpha_{t1}\,\Delta t \quad , \tag{5}$$

$$\varepsilon_2 = \alpha_{t2}\,\Delta t \quad . \tag{6}$$

By using the constitutive laws (1) and (2) we derive the following expressions for the strains in the bar layers:

$$\varepsilon_1 = \left(\frac{\sigma_1}{E_1}\right)^{\frac{1}{n_1}}, \tag{7}$$

$$\varepsilon_2 = \left(\frac{\sigma_2}{E_2}\right)^{\frac{1}{n_2}}. \tag{8}$$

Due to the fact that the layers are connected axial forces of equal magnitude, F_u, and opposite directions will appear in the layers. The magnitude of stresses in the upper and lower layers induced by F_u is found as

$$\sigma_1 = \frac{F_u}{A_1}, \tag{9}$$

$$\sigma_2 = \frac{F_u}{A_2}, \tag{10}$$

where A_1 and A_2 are the areas of the cross-sections of the layers. Having in mind that

$$A_1 = bh_1 \tag{11}$$

and

$$A_1 = bh_2 \tag{12}$$

formulas (9) and (10) are re-written as

$$\sigma_1 = \frac{F_u}{bh_1}. \tag{13}$$

$$\sigma_2 = \frac{F_u}{bh_2}. \tag{14}$$

By combining of (7), (8), (13) and (14) we obtain

$$\varepsilon_1 = \left(\frac{F_u}{bh_1E_1}\right)^{\frac{1}{n_1}}, \tag{15}$$

$$\varepsilon_2 = \left(\frac{F_u}{bh_2E_2}\right)^{\frac{1}{n_2}}. \tag{16}$$

The difference between strains in the two layers is found as

$$\Delta\varepsilon = \varepsilon_1 - \varepsilon_2 = \left(\frac{F_u}{bh_1E_1}\right)^{\frac{1}{n_1}} - \left(\frac{F_u}{bh_2E_2}\right)^{\frac{1}{n_2}} . \tag{17}$$

Besides, $\Delta\varepsilon$ can be expressed also through Δt by using (5) and (6) , i.e.

$$\Delta\varepsilon = \Delta t\left(\alpha_{t1} - \alpha_{t2}\right) . \tag{18}$$

By combining of (17) and (18), we obtain the following equation with unknown, F_u :

$$\left(\frac{F_u}{bh_1E_1}\right)^{\frac{1}{n_1}} - \left(\frac{F_u}{bh_2E_2}\right)^{\frac{1}{n_2}} = \Delta t\left(\alpha_{t1} - \alpha_{t2}\right) . \tag{19}$$

Equation (19) is solved with respect to F_u by the MatLab.

A bending moment, M , that counters the moment induced by the axial forces acts in the bi-layered bar. This bending moment can be expressed as

$$M = F_u \frac{h_1 + h_2}{2} . \tag{20}$$

The bending moment is used to analyze the curvature of the bar induced by the uniform heating. For this purpose, first, the distribution of strains along the bar thickness is written as

$$\varepsilon = \kappa\left(z - z_n\right), \tag{21}$$

where

$$-\frac{h}{2} \le z \le \frac{h}{2} . \tag{22}$$

In formula (21), z is the vertical centric axis of the bar cross-section, z_n is the coordinate of the neutral axis. Two equilibrium equations of the elementary forces in the bar cross-section are worked out for determination of the curvature and the coordinate of the neutral axis, i.e.

$$N = b\int_{-\frac{h}{2}}^{-\frac{h}{2}+h_1} \sigma_1 dz + b\int_{-\frac{h}{2}+h_1}^{\frac{h}{2}} \sigma_2 dz , \tag{23}$$

$$M = b\int_{-\frac{h}{2}}^{-\frac{h}{2}+h_1} \sigma_1 z dz + b\int_{-\frac{h}{2}+h_1}^{\frac{h}{2}} \sigma_2 z dz , \tag{24}$$

where N is the axial force (apparently, $N = 0$), the bending moment M is expressed by (20). The stresses, σ_1 and σ_2, which are involved in (23) and (24) are presented by (1) and (2), the strain is expressed by (21). Equations (23) and (24) are solved with respect to the curvature and the coordinate of the neutral axis by the MatLab.

A check-up of equations (23) and (24) is performed by using the fact that at $n_1 = n_2 = 1$ the non-linear stress-strain relationships (1) and (2) transform in the Hook's law (this means that at $n_1 = n_2 = 1$ equations (23) and

(24) should yield the curvature of a bi-layered linear-elastic bar subjected to uniform heating). In order to verify this we substitute $n_1 = n_2 = 1$ in equations (19), (23) and (24). After performing some mathematical transformations we derive from equations (23) and (24) the following formula for the curvature:

$$\kappa = \frac{M}{E_1 J_y^*}, \tag{25}$$

where

$$M = \frac{b\Delta t(\alpha_{1t} - \alpha_{2t})E_1}{\frac{1}{h_1} + \frac{1}{mh_2}} \cdot \frac{h_1 + h_2}{2}, \tag{26}$$

$$m = \frac{E_2}{E_1}. \tag{27}$$

In formula (25) J_y^* is the moment of inertia of the bi-layered bar cross-section. It should be specified that formula (25) matches the expression for the curvature of a bi-layered bar with linear-elastic behaviour under uniform heating published in (Varvak, 1997). This fact is a check-up of the curvature analysis developed in the present paper. The case of a bar whose layers are continuously inhomogeneous in longitudinal direction is also analyzed here. In this case the material properties of the layers are distributed continuously along the bar length, i.e.

$$E_1 = E_1(x), \tag{28}$$

$$E_2 = E_2(x), \tag{29}$$

$$n_1 = n_1(x), \tag{30}$$

$$n_2 = n_2(x), \tag{31}$$

$$\alpha_{t1} = \alpha_{t1}(x), \tag{32}$$

$$\alpha_{t2} = \alpha_{t2}(x), \tag{33}$$

where

$$0 \le x \le l. \tag{34}$$

In formulas (28) – (34) x is the longitudinal centroidal axis of the inhomogeneous bar under consideration. The prolongations, Δl_{t1} and Δl_{t2}, of the upper and lower bar layers (if they are not connected) can be found as

$$\Delta l_{t1} = \int_0^l \alpha_{t1} \Delta t \, dx, \tag{35}$$

$$\Delta l_{t2} = \int_0^l \alpha_{t2} \Delta t \, dx. \tag{36}$$

Since the two layers are continuously inhomogeneous in longitudinal direction, the strains, ε_1 and ε_2, are continuous functions of x, i.e.

$$\varepsilon_1(x) = \alpha_{t1} \Delta t \ , \tag{37}$$

$$\varepsilon_2(x) = \alpha_{t2} \Delta t \ . \tag{38}$$

For the bi-layered inhomogeneous bar under uniform heating equation (19) takes the form

$$\left[\frac{F_u}{bh_1E_1(x)}\right]^{\frac{1}{n_1(x)}}-\left[\frac{F_u}{bh_2E_2(x)}\right]^{\frac{1}{n_2(x)}}=\Delta t\left[\alpha_{t1}(x)-\alpha_{t1}(x)\right]. \tag{39}$$

Equation (39) is used for deriving F_u at different values of x. Then equations (23) and (24) are applied for obtaining the uniform heating induced bar curvature at x varying in the interval $[0;l]$.

Conclusion

The paper describes a theoretical approach for analysis the curvature in a bi-layered bar subjected to uniform heating. The bar has non-linear elastic mechanical behaviour. The cross-section of the bar is a rectangle. A power law stress-strain relationship is applied for dealing with material non-linearity of the bar layers. First, the case of a bar with homogeneous layers is analyzed. Equations for deriving of the axial forces and the bending moment are constituted by considering the strains in the bar layers induced by uniform heating (in particular, we use here the circumstance that the strains along the contact line between the two layers are the same). The equations obtained are non-linear. After that the so derived axial forces and the bending moment are inserted in the equations for determining the curvature and the neutral axis coordinate of the bar (these equations are composed by analyzing the equilibrium of the elementary forces in the bar cross-section). It should be specified here that due to the fact that the bar is made by layers having different properties the neutral axis does not pass via the cross-section centre. As a result of this there are two unknowns (curvature and coordinate of the neutral axis) in the equations of equilibrium. The equations for determining the curvature and the coordinate of the neutral axis are checked-up by using the fact that at $n_1 = n_2 = 1$ the non-linear stress-strain relationship applied here transform into the Hook's law. Therefore, we prove that by substituting of $n_1 = n_2 = 1$ in the equations for the curvature and the coordinate of the neutral axis composed in this paper the equations yield an expression for the curvature of a bi-layered bar of linear-elastic behaviour subjected to uniform heating that is a match of a known solution for the curvature published in the scientific literature. A consideration is given also to the problem of determination of the uniform heating induced curvature in a non-linear elastic bar made of two layers which are continuously inhomogeneous in longitudinal direction. The bar has a rectangular cross-section. Due to the fact that the layers are inhomogeneous the strains change continuously along the bar length. Equations for determination the axial forces and the bending moments in various sections of the bar are composed. These axial forces and bending moments are substituted in the equations of equilibrium to determine the bar curvature. From practical view-point, the curvature and the coordinate of neutral axis determined by the approach developed in the present paper can be applied for studying the stressed and strained state of bi-layered bars exhibiting material non-linearity under uniform heating. Besides, the approach can be refined with purpose of analyzing curvature of bars having an arbitrary number of non-linear elastic layers subjected to uniform heating.

Recommendations

Determination of the curvature of bi-layered bars under uniform heating by applying various non-linear stress-strain constitutive laws can be recommended as a future task.

Scientific Ethics Declaration

The author declares that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the author.

Acknowledgements or Notes

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References

Dolgov, N. (2005). Determination of Stresses in a two-layer coating. *Strength of Materials, 37, 422-431.*

Dolgov, N. (2016). Analytical methods to determine the stress state in the substrate–coating system under mechanical loads. *Strength of Materials, 48*(5), *658-667.*

Gasik, M. M. (2010). Functionally graded materials: bulk processing techniques. *International Journal of Materials and Product Technology, 39*(1), *20-29.*

Hirai, T., & Chen, L. (1999). Recent and prospective development of functionally graded materials in Japan. *Mater Sci. Forum, 308-311,* 509-514.

Kaul, A. B. (2014). Two-dimensional layered materials: Structure, properties, and prospects for device applications, *Journal of Material Research*, *29*(03), *348-361*.

Kim, J. S., Paik, K.W., & Oh, S. H. (1999). The multilayer-modified stoney's formula for laminated polymer composites on a silicon substrate. *Journal of Applied Physics, 86,* 5474–5479.

Lloyd, S.J., & Molina-Aldareguia, J. M. (2003), Multilayered materials: a palette for the materials artist. *Phil. Trans. R. Soc. Lond. A*, *361*(1813), 2931-294.

Radhika, N., Sasikumar, J., Sylesh, J. L., & Kishore, R. (2020). Dry reciprocating wear and frictional behaviour of B4C reinforced functionally graded and homogenous aluminium matrix composites. *Journal of Materials Research and Technology*, *9*(2), 1578-1592.

Rizov, V.I. (2017). Delamination analysis of a layered elastic-plastic beam. *International Journal of Structural Integrity, 8,* 516-529.

Rizov, V.I. (2018). Analysis of cylindrical delamination cracks in multilayered functionally graded non-linear elastic circular shafts under combined loads. *Frattura ed Integrità Strutturale, 12*(46)*, 158-17.*

Rizov, V.I. (2019). Influence of material inhomogeneity and non-linear mechanical behavior of the material on delamination in multilayered beams. *Frattura ed Integrita Strutturale, 13*(47), *468-481.*

Rzhanitsyn, A.R. (1986). *Built-up ars and plates*. Stroyizdat.

Varvak, P. (1997). *Strength of materials*. Science.

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