The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM)
ISSN: 2602-3199

The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM), 2023

Volume 25, Pages 168-186

ICBAST 2023: International Conference on Basic Sciences and Technology

G**β-Continuous and G**β-Irresolute Mappings in Topological Spaces

Raja Mohammad Latif
Prince Mohammad Bin Fahd University

Abstract: Topology being somehow very recent in nature but has got tremendous applications over almost all other fields. Theoretical or fundamental topology is a bit dry but the application part is what drives crazy once we get used. Topology has applications in various fields of Science and Technology, like applications to Biology, Robotics, GIS, Engineering, Computer Sciences. Topology though being a part of mathematics but it has influenced the whole world with so strong effects and incredible applications. The concept of continuity is fundamental in large parts of contemporary mathematics. In the nineteenth century, precise definitions of continuity were formulated for functions of a real or complex variable, enabling mathematicians to produce rigorous proofs of fundamental theorems of real and complex analysis, such as the Intermediate Value Theorem, Taylor's Theorem, the Fundamental Theorem of Calculus, and Cauchy's Theorem. In the early years of the Twentieth Century, the concept of continuity was generalized so as to be applicable to functions between metric spaces, and subsequently to functions between topological spaces. Topology is an area of mathematics concerned with the properties of space that are preserved under continuous deformations including stretching and bending but not tearing. In 2023, Dr. T. Delcia and M. S, Thillai introduced a new type of closed sets called g**β-closed sets and investigated their basic properties including their relationship with already existing concepts in Topological Spaces. In this paper, we introduce g**β-continuous function, g**β-irresolute function, g**β-open function, g**β-closed function, pre-g**β-open function, and pre-g**β-closed function, and investigate properties and characterizations of these new types of mappings in topological spaces.

Keywords: Topological space, g**β-closed set, g**β-continuous function, g**β-irresolute function, g**β-open function

1. Introduction

Introducing new versions of open sets in a topological space which may acquire either weaker or stronger properties is often studied. The first attempt was done by Levine [13], where he introduced the concepts of semi-open set, semi-closed set, and semi-continuity of a function. In 2023, Dr. T. Delcia and M. S, Thillai introduced a new type of closed sets called g**β-closed sets and investigated their basic properties including their relationship with already existing concepts in Topological Spaces.

In this paper, we introduce g**β-continuous function, g**β-irresolute function, g**β-open function, g**β-closed function, pre-g**β-open function, and pre-g**β-closed function, and investigate properties and characterizations of these new types of mappings in topological spaces.

2. BASIC PROPERTIES AND APPLICATIONS OF $g^{**}\beta$-OPEN SETS

In this section, we shall define the concept of $g^{**}\beta$-open set and determine its connection to the classical open set and characterize the concepts of $g^{**}\beta$-open sets.

Definition 2.1. A subset A of a topological space (X,τ) is named generalized closed $(g\text{-}closed)$ closed if $Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in X. The complement of g-closed set is called g-open in X.

Definition 2.2. A subset A of a topological space (X,τ) is called a generalized star closed set (briefly g^*-closed) closed set if $Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is g-open in X. The complement of $g^*\text{-}closed$ set is called $g^*\text{-}open$ set.

Definition 2.3. A subset A of a topological space (X,τ) is called a generalized star β $(g*\beta\text{-}closed)$ closed if $\beta Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is $g*\text{-}open$ in X. The complement of $g*\beta\text{-}closed$ set is called $g*\beta-$open set.

Definition 2.4. A subset A of a topological space (X,τ) is called a generalized star star β $(g^{**}\beta\text{-}closed)$ closed if $gCl(A)\subseteq U$ whenever $A\subseteq U$ and U is $g^*\beta\text{-}open$ in X. The complement of $g^{**}\beta\text{-}closed$ set is called $g^{**}\beta-$open set. The collection of all $g^{**}\beta$-open $(g^{**}\beta$-closed$)$ subsets of (X,τ) is denoted by $g^{**}\beta\text{-}O(X,\tau)$ $(g^{**}\beta\text{-}C(X,\tau))$.

Theorem 2.5. Every closed (resp. open) set is $g^{**}\beta$-closed $(\text{resp.}\, g^{**}\beta$-open$)$ set.

Theorem 2.6. If A and B are $g^{**}\beta-$closed $(\text{resp.}\, g^{**}\beta-$open$)$ sets in X, then $A\cup B$ $(\textit{resp. } A\cap B)$ is $g^{**}\beta$-closed $(\text{resp.}\, g^{**}\beta-$open$)$ in X.

Theorem 2.7. Arbitrary intersection (union) of $g^{**}\beta-$closed $(\text{resp.}\, g^{**}\beta-$open$)$ sets is $g^{**}\beta-$closed $(\text{resp.}\, g^{**}\beta-$open$)$ set in X.

Proof. Let $\Gamma=\{A_i : i\in I\}\subseteq g^{**}\beta-C(X,\tau)$. Let U be a $g^*\beta-$open set in X such that $A_i\subseteq U$ for each $i\in I$. Hence, $gC(A_i)\subseteq U$ for each $i\in I$. Therefore, $gCl(\bigcap\Gamma)=gCl(\bigcap_{i\in I}A_i\Gamma)\subseteq\bigcap_{i\in I}gCl(A)_i\subseteq U$. This implies $gCl(\bigcap\Gamma)=gCl(\bigcap_{i\in I}A_i\Gamma)\subseteq U$, U is $g^{**}\beta-$open in X. Hence $\bigcap\Gamma=\bigcap_{i\in I}A_i$ is $g^{**}\beta-$closed in X. Thus arbitrary intersection of $g^{**}\beta-$closed sets is $g^{**}\beta-$closed set in X. Now by using this result and complements we conclude that arbitrary union of $g^{**}\beta-$open sets is $g^{**}\beta-$open set in X.

Definition 2.8. Let (X,τ) be a topological space and $B\subseteq X$. We define the $g^{**}\beta$-closure of B (briefly $g^{**}\beta\text{-}Cl(B)$) to be the intersection of all $g^{**}\beta$-closed sets containing B which is denoted by $g^{**}\beta\text{-}Cl(B)=\bigcap\{A : B\subseteq A \text{ and } A\in g^{**}\beta\text{-}C(X,\tau).\}$. We note that $g^{**}\beta\text{-}Cl(B)$ is the smallest $g^{**}\beta\text{-}closed$ set containing B.

Definition 2.9. Let (X,τ) be any topological space and B be a subset of X. A point p of X is called a $g^{**}\beta\text{-}interior$ point of B, if there exists a $g^{**}\beta\text{-}open$ set G such that $p\in G\subseteq B$. The set of all $g^{**}\beta\text{-}interior$ points of B is said to be $g^{**}\beta\text{-}interior$ of B (briefly $g^{**}\beta\text{-}Int(B)$) and it is defined by $g^{**}\beta\text{-}Int(B)=\bigcup\{A : A\subseteq B \text{ and } A\in g^{**}\beta\text{-}O(X,\tau)\}$.

Definition 2.10. Let ¥ be a subset of a topological space (X,τ) and let $x \in X$. We say that ¥ is $g^{**}\beta$-*neighborhood* of x, if there is a $g^{**}\beta$-*open* set U such that $x \in U \subseteq$ ¥.

Proposition2.11. If U and V are sets in a topological space (X,τ), then

(1) $g^{**}\beta\text{-}Int(\phi)=\phi$. (2) $g^{**}\beta\text{-}Int(X)=X$. (3) $g^{**}\beta\text{-}Int(U)\subseteq U$.

(4) $U \subseteq V \Rightarrow g^{**}\beta\text{-}Int(U) \subseteq g^{**}\beta\text{-}Int(V)$.

Proposition2.12. Let G be any subset of a topological space (X,τ). Then $x \in g^{**}\beta\text{-}Cl(A)$ if and only if for every $g^{**}\beta$-*open* set U containing x, $U \cap G \neq \phi$.

Proposition2.13. For any subset U of topological space (X,τ), $g^{**}\beta\text{-}Int(U) \subseteq U \subseteq g^{**}\beta\text{-}Cl(U)$.

Definition 2.14. Let A be a subset of a topological space (X,τ). A point $x \in A$ is said to be a $g^{**}\beta$-*limit* point of A if for each $g^{**}\beta$-*open* set U containing x, $U \cap (A-\{x\}) \neq \phi$. The set of all $g^{**}\beta$-*limit* points of A is called the $g^{**}\beta$-*derived* set of A and is denoted by $g^{**}\beta\text{-}D(A)$.

Theorem 2.15. For any subset A of a topological space X, $g^{**}\beta\text{-}Cl(A) = A \cup [g^{**}\beta\text{-}D(A)]$.

Proof. Since $g^{**}\beta\text{-}D(A) \subseteq g^{**}\beta\text{-}Cl(A)$. $A \cup [g^{**}\beta\text{-}D(A)] \subseteq g^{**}\beta\text{-}Cl(A)$. On the other hand, let $x \in g^{**}\beta\text{-}Cl(A)$. If $x \in A$, then the proof is complete. If $x \notin A$, each $g^{**}\beta$-*open* set U containing x intersects A at a point distinct from x, so $x \in g^{**}\beta\text{-}D(A)$. Thus, $g^{**}\beta\text{-}Cl(A) \subseteq [A \cup (g^{**}\beta\text{-}D(A))]$.

Corollary 2.16. A subset A of a space X is $g^{**}\beta$-*closed* if and only if it contains the set of all of its $g^{**}\beta$-*limit* points.

Theorem 2.17. For subsets A, B of a space X, the following statements are true:

(1) $g^{**}\beta\text{-}Int(A)$ is the largest $g^{**}\beta$-*open* set contained in A;
(2) A is $g^{**}\beta$-*open* if and only if $A = g^{**}\beta\text{-}Int(A)$.
(3) $g^{**}\beta\text{-}Int[g^{**}\beta\text{-}Int(A)] = g^{**}\beta\text{-}Int(A)$;
(4) $g^{**}\beta\text{-}Int(A) = [A-(g^{**}\beta\text{-}D(X-A))]$;
(5) $[X-(g^{**}\beta\text{-}Cl(A))] = g^{**}\beta\text{-}Int(X-A)$;
(6) $[X-(g^{**}\beta\text{-}Int(A))] = g^{**}\beta\text{-}Cl(X-A)$;
(7) $[g^{**}\beta\text{-}Int(A)] \cup [g^{**}\beta\text{-}Int(B)] \subseteq g^{**}\beta\text{-}Int(A \cup B)$;
(8) $g^{**}\beta\text{-}Int(A \cap B) = [g^{**}\beta\text{-}Int(A)] \cap [g^{**}\beta\text{-}Int(B)]$;

(4) If $x \in [A-(g^{**}\beta\text{-}D(X-A))]$, then $x \notin g^{**}\beta\text{-}D(X-A)$ and so there exists a $g^{**}\beta$-*open* set U containing x such that $U \cap (X-A)=\phi$. Then, $x \in U \subseteq A$ and hence $x \in g^{**}\beta\text{-}Int(A)$, that is, $[A-(g^{**}\beta\text{-}D(X-A))] \subseteq g^{**}\beta\text{-}Int(A)$. On the other hand, if $x \in g^{**}\beta\text{-}Int(A)$, then $x \notin g^{**}\beta\text{-}D(X-A)$

since $g^{**}\beta\text{-}Int(A)$ is $g^{**}\beta\text{-}open$ and $\left[\left(g^{**}\beta\text{-}Int(A)\right)\cap(X-A)\right]=\phi$. Hence, $g^{**}\beta\text{-}Int(A)=\left[A-\left(g^{**}\beta\text{-}D(X-A)\right)\right]$.

(6) $X-\left[g^{**}\beta\text{-}Int(A)\right]=X-\left[A-\left(g^{**}\beta\text{-}D(X-A)\right)\right]=(X-A)\cup\left[g^{**}\beta\text{-}D(X-A)\right]=$ $g^{**}\beta\text{-}Cl(X-A)$.

Theorem 2.18. Let (X,τ) be a topological space and $A, B\subseteq X$. Then the following statements are true:

(1) $x\in g^{**}\beta\text{-}Cl(A)$ if and only if for every $g^{**}\beta\text{-}open$ subset U containing x, $U\cap A\neq\phi$.

(2) $A\subseteq B$ implies that $g^{**}\beta\text{-}Cl(A)\subseteq g^{**}\beta\text{-}Cl(B)$.

(3) A is $g^{**}\beta\text{-}closed$ if and only if $g^{**}\beta\text{-}Cl(A)=A$.

(4) $g^{**}\beta\text{-}Cl\left[g^{**}\beta\text{-}Cl(A)\right]=g^{**}\beta\text{-}Cl(A)$.

(5) $\left[g^{**}\beta\text{-}Cl(A)\right]\cup\left[g^{**}\beta\text{-}Cl(B)\right]=g^{**}\beta\text{-}Cl(A\cup B)$.

(6) $g^{**}\beta\text{-}Int(X-A)=X-\left[g^{**}\beta\text{-}Cl(A)\right]$.

(7) $g^{**}\beta\text{-}Cl(X-A)=X-\left[g^{**}\beta\text{-}Int(A)\right]$.

Definition 2.19. $g^{**}\beta\text{-}Bd(A)=A-\left[g^{**}\beta\text{-}Int(A)\right]$ is said to be the $g^{**}\beta\text{-}border$ of A.

Theorem 2.20. For a subset A of a space X, the following statements hold:

(1) $Bd(A)\subseteq g^{**}\beta\text{-}Bd(A)$ where $Bd(A)$ denotes the border of A;

(2) $A=g^{**}\beta\text{-}Int(A)\cup g^{**}\beta\text{-}Bd(A)$;

(3) $\left[g^{**}\beta\text{-}Int(A)\right]\cap\left[g^{**}\beta\text{-}Bd(A)\right]=\phi$;

(4) A is a $g^{**}\beta\text{-}open$ set if and only if $g^{**}\beta\text{-}Bd(A)=\phi$;

(5) $g^{**}\beta\text{-}Bd\left[g^{**}\beta\text{-}Int(A)\right]=\phi$;

(6) $g^{**}\beta\text{-}Int\left[g^{**}\beta\text{-}Bd(A)\right]=\phi$;

(7) $g^{**}\beta\text{-}Bd\left[g^{**}\beta\text{-}Bd(A)\right]=g^{**}\beta\text{-}Bd(A)$;

(8) $g^{**}\beta\text{-}Bd(A)=A\cap\left[g^{**}\beta\text{-}Cl(X-A)\right]$;

(9) $g^{**}\beta\text{-}Bd(A)=g^{**}\beta\text{-}D(X-A)$.

Proof. (6) If $x\in g^{**}\beta\text{-}Int\left[g^{**}\beta\text{-}Bd(A)\right]$, then $x\in g^{**}\beta\text{-}Bd(A)$. On the other hand, since $g^{**}\beta\text{-}Bd(A)\subseteq A$, $x\in g^{**}\beta\text{-}Int\left[g^{**}\beta\text{-}Bd(A)\right]\subseteq g^{**}\beta\text{-}Int(A)$. Therefore, we get $x\in\left[g^{**}\beta\text{-}Int(A)\right]\cap\left[g^{**}\beta\text{-}Bd(A)\right]$, which contradicts (3). Thus, $g^{**}\beta\text{-}Int\left[g^{**}\beta\text{-}Bd(A)\right]=\phi$.

(8) $g^{**}\beta\text{-}Bd(A)=A-\left[g^{**}\beta\text{-}Int(A)\right]=A-\left[X-\left(g^{**}\beta\text{-}Cl(X-A)\right)\right]=A\cap\left[g^{**}\beta\text{-}Cl(X-A)\right]$.

(9) $g^{**}\beta\text{-}Bd(A)=A-\left[g^{**}\beta\text{-}Int(A)\right]=A-\left[A-\left(g^{**}\beta\text{-}D(X-A)\right)\right]=g^{**}\beta\text{-}D(X-A)$.

Definition 2.21. $g^{**}\beta\text{-}Fr(A)=\left[g^{**}\beta\text{-}Cl(A)\right]-\left[g^{**}\beta\text{-}Int(A)\right]$ is said to be the $g^{**}\beta$-frontier of A.

Theorem 2.22. For a subset A of a space X, the following statements hold:

(1) $Fr(A)\subseteq g^{**}\beta\text{-}Fr(A)$ where $Fr(A)$ denotes the frontier of A;

(2) $g^{**}\beta\text{-}Cl(A)=\left[g^{**}\beta\text{-}Int(A)\right]\cup\left[g^{**}\beta\text{-}Fr(A)\right]$;

(3) $\left[g^{**}\beta\text{-}Int(A)\right]\cap\left[g^{**}\beta\text{-}Fr(A)\right]=\phi$;

(4) $g^{**}\beta\text{-}Bd(A)\subseteq g^{**}\beta\text{-}Fr(A)$;

(5) $g^{**}\beta\text{-}Fr(A)=\left[g^{**}\beta\text{-}Bd(A)\right]\cup\left[g^{**}\beta\text{-}D(A)\right]$;

(6) A is a $g^{**}\beta\text{-}open$ set if and only if $g^{**}\beta\text{-}Fr(A)=g^{**}\beta\text{-}D(A)$;

(7) $g^{**}\beta\text{-}Fr(A)=\left[g^{**}\beta\text{-}Cl(A)\right]\cap\left[g^{**}\beta\text{-}Cl(X-A)\right]$;

(8) $g^{**}\beta\text{-}Fr(A)=g^{**}\beta\text{-}Fr(X-A)$;

(9) $g^{**}\beta\text{-}Fr(A)$ is $g^{**}\beta\text{-}closed$;

(10) $g^{**}\beta\text{-}Fr\left[g^{**}\beta\text{-}Fr(A)\right]\subseteq g^{**}\beta\text{-}Fr(A)$;

(11) $g^{**}\beta\text{-}Fr\left[g^{**}\beta\text{-}Int(A)\right]\subseteq g^{**}\beta\text{-}Fr(A)$;

(12) $g^{**}\beta\text{-}Fr\left[g^{**}\beta\text{-}Cl(A)\right]\subseteq g^{**}\beta\text{-}Fr(A)$;

(13) $g^{**}\beta\text{-}Int(A)=A-\left[g^{**}\beta\text{-}Fr(A)\right]$.

Proof. (2) $\left[g^{**}\beta\text{-}Int(A)\right]\cup\left[g^{**}\beta\text{-}Fr(A)\right]=$
$g^{**}\beta\text{-}Int(A)\cup\left[g^{**}\beta\text{-}Cl(A)-g^{**}\beta\text{-}Int(A)\right]=g^{**}\beta\text{-}Cl(A)$

(3) $\left[g^{**}\beta\text{-}Int(A)\right]\cap\left[g^{**}\beta\text{-}Fr(A)\right]=\left[g^{**}\beta\text{-}Int(A)\right]\cap\left[\left(g^{**}\beta\text{-}Cl(A)\right)-\left(g^{**}\beta\text{-}Int(A)\right)\right]=\phi.$

(5) $Since \left[g^{**}\beta\text{-}Int(A)\right]\cup\left[g^{**}\beta\text{-}Fr(A)\right]=\left[g^{**}\beta\text{-}Int(A)\right]\cup\left[g^{**}\beta\text{-}Bd(A)\right]\cup\left[g^{**}\beta\text{-}D(A)\right]$,
$g^{**}\beta\text{-}Fr(A)=\left[g^{**}\beta\text{-}Bd(A)\right]\cup\left[g^{**}\beta\text{-}D(A)\right]$.

(7) $g^{**}\beta\text{-}Fr(A)=\left[g^{**}\beta\text{-}Cl(A)\right]-\left[g^{**}\beta\text{-}Int(A)\right]=\left[g^{**}\beta\text{-}Cl(A)\right]\cap\left[g^{**}\beta\text{-}Cl(X-A)\right]$.

(9) $g^{**}\beta\text{-}Cl\left[g^{**}\beta\text{-}Fr(A)\right]=g^{**}\beta\text{-}Cl\left[\left(g^{**}\beta\text{-}Cl(A)\right)\cap\left(g^{**}\beta\text{-}Cl(X-A)\right)\right]\subseteq$
$g^{**}\beta\text{-}Cl\left[g^{**}\beta\text{-}Cl(A)\right]\cap g^{**}\beta\text{-}Cl\left[g^{**}\beta\text{-}Cl(X-A)\right]$
$=\left[g^{**}\beta\text{-}Cl(A)\right]\cap\left[g^{**}\beta\text{-}Cl(X-A)\right]=g^{**}\beta\text{-}Fr(A)$. Hence $g^{**}\beta\text{-}Fr(A)$ is $g^{**}\beta\text{-}closed$.

(10) $g^{**}\beta\text{-}Fr\left[g^{**}\beta\text{-}Fr(A)\right]=g^{**}\beta\text{-}Cl\left[g^{**}\beta\text{-}Fr(A)\right]\cap g^{**}\beta\text{-}Cl\left[X-\left(g^{**}\beta\text{-}Fr(A)\right)\right]\subseteq$
$g^{**}\beta\text{-}Cl\left[g^{**}\beta\text{-}Fr(A)\right]=g^{**}\beta\text{-}Fr(A)$.

(12) $g^{**}\beta\text{-}Fr\left[g^{**}\beta\text{-}Cl(A)\right]=g^{**}\beta\text{-}Cl\left[g^{**}\beta\text{-}Cl(A)\right]-g^{**}\beta\text{-}Int\left[g^{**}\beta\text{-}Cl(A)\right]=$
$\left[g^{**}\beta\text{-}Cl(A)\right]-\left[g^{**}\beta\text{-}Int\left(g^{**}\beta\text{-}Cl(A)\right)\right]\subseteq$
$\left[\left(g^{**}\beta\text{-}Cl(A)\right)-\left(g^{**}\beta\text{-}Int(A)\right)\right]=g^{**}\beta\text{-}Fr(A).$

(13) $A-\left[g^{**}\beta\text{-}Fr(A)\right]=A-\left[\left(g^{**}\beta\text{-}Cl(A)\right)-\left(g^{**}\beta\text{-}Int(A)\right)\right]=g^{**}\beta\text{-}Int(A)$.

$3-8$: CHARACTERIZATIONS OF MAPPINGS

The purpose of this part is to introduce $g^{**}\beta\text{-}continuous$, $g^{**}\beta\text{-}irresolute$, $g^{**}\beta\text{-}open$, $g^{**}\beta\text{-}closed$, $pre\text{-}g^{**}\beta\text{-}open$, and $pre\text{-}g^{**}\beta\text{-}closed$ functions and explore properties and characterizations of these functions.

3. $g^{**}\beta$-CONTINUOUS FUNCTIONS

The purpose of this section is to investigate the properties and characterizations of $g^{**}\beta$-*continuous* functions.

Definition 3.1. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be $g^{**}\beta$-*continuous* if $f^{-1}(V)\in g^{**}\beta\text{-}O(X,\tau)$ for every $V\in\sigma$.

Theorem 3.2. Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then the following statements are equivalent:

(1) f is $g^{**}\beta$-*continuous*.

(2) The inverse image of each closed set in Y is a $g^{**}\beta$-*closed* set in X;

(3) $g^{**}\beta\text{-}Cl\left[f^{-1}(V)\right]\subseteq f^{-1}\left[Cl(V)\right]$, for every $V\subseteq Y$;

(4) $f\left[g^{**}\beta\text{-}Cl(U)\right]\subseteq Cl\left[f(U)\right]$, for every $U\subseteq X$;

(5) For any point $x\in X$ and any open set V of Y containing $f(x)$, there exists $U\in g^{**}\beta\text{-}O(X,\tau)$ such that $x\in U$ and $f(U)\subseteq V$;

(6) $g^{**}\beta\text{-}Bd\left[f^{-1}(V)\right]\subseteq f^{-1}\left[g^{**}\beta\text{-}d(V)\right]$, for every $V\subseteq Y$;

(7) $f\left[g^{**}\beta\text{-}D(U)\right]\subseteq Cl\left[f(U)\right]$, for every $U\subseteq X$;

(8) $f^{-1}\left[Int(V)\right]\subseteq g^{**}\beta\text{-}Int\left[f^{-1}(V)\right]$, for every $V\subseteq Y$;

Proof. $(1)\Rightarrow(2)$: Let $F\subseteq Y$ be closed. Since f is $g^{**}\beta$-*continuous*, $f^{-1}(Y-F)=X-f^{-1}(F)$ is $g^{**}\beta$-*open*. Therefore, $f^{-1}(F)$ is $g^{**}\beta$-*closed* in X.

$(2)\Rightarrow(3)$: Since $Cl(V)$ is closed for every $V\subseteq Y$, then $f^{-1}\left[Cl(V)\right]$ is $g^{**}\beta$-*closed*. Therefore $f^{-1}\left[Cl(V)\right]=g^{**}\beta\text{-}Cl\left[f^{-1}(Cl(V))\right]\supseteq g^{**}\beta\text{-}Cl\left[f^{-1}(V)\right]$.

$(3)\Rightarrow(4)$: Let $U\subseteq X$ and $f(U)=V$. Then $g^{**}\beta\text{-}Cl\left[f^{-1}(V)\right]\subseteq f^{-1}\left[Cl(V)\right]$. Thus $g^{**}\beta\text{-}Cl(U)\subseteq g^{**}\beta\text{-}Cl\left[f^{-1}(f(U))\right]\subseteq f^{-1}\left[Cl(f(U))\right]$ and $f\left[g^{**}\beta\text{-}Cl(U)\right]\subseteq Cl\left[f(U)\right]$.

$(4)\Rightarrow(2)$: Let $W\subseteq Y$ be a closed set, and $U=f^{-1}(W)$. Then $f\left[g^{**}\beta\text{-}Cl(U)\right]\subseteq Cl\left[f(U)\right]$ $=Cl\left[f(f^{-1}(W))\right]\subseteq Cl(W)=W$. Thus $g^{**}\beta\text{-}Cl(U)\subseteq f^{-1}\left[f(g^{**}\beta\text{-}Cl(U))\right]\subseteq f^{-1}(W)=U$. So U is $g^{**}\beta$-*closed*.

$(2)\Rightarrow(1)$: Let $V\subseteq Y$ be an open set. Then $Y-V$ is closed. Then $f^{-1}(Y-V)=X-f^{-1}(V)$ is $g^{**}\beta$-*closed* in X and hence $f^{-1}(V)$ *is* $g^{**}\beta$-*open in* X.

$(1)\Rightarrow(5)$: Let $f:(X,\tau)\to(Y,\sigma)$ be $g^{**}\beta$-*continuous*. For any $x\in X$ and any open set V of Y containing $f(x)$, $U=f^{-1}(V)\in g^{**}\beta\text{-}O(X,\tau)$, and $f(U)=f\left[f^{-1}(V)\right]\subseteq V$.

$(5)\Rightarrow(1)$: Let $V\in\sigma$. We prove $f^{-1}(V)\in g^{**}\beta\text{-}O(X,\tau)$. Let $x\in f^{-1}(V)$. Then $f(x)\in V$ and there exists $U\in g^{**}\beta\text{-}O(X,\tau)$ such that $x\in U$ and $f(x)\in f(U)\subseteq V$. Hence $x\in U\subseteq f^{-1}\left[f(U)\right]\subseteq f^{-1}(V)$. It shows that $f^{-1}(V)$ is a $g^{**}\beta$-*neighborhood* of each of its points. Therefore $f^{-1}(V)\in g^{**}\beta\text{-}O(X,\tau)$.

$(6)\Rightarrow(8)$: Let $V\subseteq Y$. Then by hypothesis, $g^{**}\beta\text{-}Bd\left[f^{-1}(V)\right]\subseteq f^{-1}\left[Bd(V)\right]$

$\Rightarrow f^{-1}(V)-\left[g^{**}\beta\text{-}Int(f^{-1}(V))\right]\subseteq f^{-1}\left[V-Int(V)\right]=f^{-1}(V)-f^{-1}\left[Int(V)\right]$

$\Rightarrow f^{-1}\left[Int(V)\right]\subseteq g^{**}\beta\text{-}Int\left[f^{-1}(V)\right]$.

$(8)\Rightarrow(6)$: Let $V\subseteq Y$. Then by hypothesis, $f^{-1}\left[Int(V)\right]\subseteq g^{**}\beta\text{-}Int\left[f^{-1}(V)\right]$

$\Rightarrow\ f^{-1}(V)-\left[g^{**}\beta\text{-}Int\left(f^{-1}(V)\right)\right]\subseteq f^{-1}(V)-f^{-1}\left[Int(V)\right]=f^{-1}\left[V-Int(V)\right]$

$\Rightarrow g^{**}\beta\text{-}Bd\left[f^{-1}(V)\right]\subseteq f^{-1}\left[Bd(V)\right]$.

$(1)\Rightarrow(7)$: It is obvious, since f *is* $g^{**}\beta$-*continuous* and by (4) $f\left[g^{**}\beta\text{-}Cl(U)\right]\subseteq Cl\left[f(U)\right]$ for each $U\subseteq X$. So $f\left[g^{**}\beta\text{-}D(U)\right]\subseteq Cl\left[f(U)\right]$.

$(7)\Rightarrow(1)$: Let $U\subseteq Y$ be an open set, $V=Y-U$ and $f^{-1}(V)=W$. Then by hypothesis $f\left[g^{**}\beta\text{-}D(W)\right]\subseteq Cl\left[f(W)\right]$. Thus $f\left[g^{**}\beta\text{-}D\left(f^{-1}(V)\right)\right]\subseteq Cl\left[f\left(f^{-1}(V)\right)\right]\subseteq Cl(V)=V$. Then $g^{**}\beta\text{-}D\left[f^{-1}(V)\right]\subseteq f^{-1}(V)$ and $f^{-1}(V)$ *is* $g^{**}\beta$-*closed*. Therefore, f is $g^{**}\beta$-*continuous*.

$(1)\Rightarrow(8)$: Let $V\subseteq Y$. Then $f^{-1}\left[Int(V)\right]$ is $g^{**}\beta$-*open in* X. Thus $f^{-1}\left[Int(V)\right]=$ $g^{**}\beta\text{-}Int\left[f^{-1}\left(Int(V)\right)\right]\subseteq$ $g^{**}\beta\text{-}Int\left[f^{-1}(V)\right]$. Therefore $f^{-1}\left[Int(V)\right]\subseteq g^{**}\beta\text{-}Int\left[f^{-1}(V)\right]$.

$(8)\Rightarrow(1)$: Let $V\subseteq Y$ be an open set. Then $f^{-1}(V)=f^{-1}\left[Int(V)\right]\subseteq g^{**}\beta\text{-}Int\left[f^{-1}(V)\right]$. Therefore, $f^{-1}(V)$ *is* $g^{**}\beta$-*open*. Hence f *is* $g^{**}\beta$-*continuous*.

In the next Theorem, $\#g^{**}\beta\text{-}c.$ denotes the set of points x of X for which a function $f:(X,\tau)\to(Y,\sigma)$ is not $g^{**}\beta$-*continuous*.

Theorem 3.3. $\#g^{**}\beta\text{-}c.$ is identical with the union of the $g^{**}\beta$-frontiers of the inverse images of $g^{**}\beta$-*open* sets containing $f(x)$.

Proof. Suppose that f is not $g^{**}\beta$-*continuous* at a point x of X. Then there exists an open set $V\subseteq Y$ containing $f(x)$ such that $f(U)$ is not a subset of V for every $U\in g^{**}\beta\text{-}O(X,\tau)$. containing x. Hence, we have $U\cap f^{-1}\left(X-f^{-1}(V)\right)\neq\phi$ for every $U\in g^{**}\beta\text{-}O(X,\tau)$ containing x. It follows that $x\in\left[g^{**}\beta\text{-}Cl\left(X-f^{-1}(V)\right)\right]$. We also have $x\in f^{-1}(V)\subseteq g^{**}\beta\text{-}Cl\left[f^{-1}(V)\right]$. This means that $x\in g^{**}\beta\text{-}Fr\left[f^{-1}(V)\right]$. Now, let f be $g^{**}\beta$-*continuous* at $x\in X$ and $V\subseteq Y$ any open set containing $f(x)$. Then, $x\in f^{-1}(V)$ is a $g^{**}\beta$-*open* set of X. Thus. $x\in g^{**}\beta\text{-}Int\left[f^{-1}(V)\right]$ and therefore $x\notin g^{**}\beta\text{-}Fr\left[f^{-1}(V)\right]$ for every open set V containing $f(x)$.

Remarks 3.4. (1) Every $g^{**}\beta$-*continuous* function is continuous, but the converse may not be true.

(2) If a function $f:(X,\tau)\to(Y,\sigma)$ is $g^{**}\beta$-*continuous* and a function $g:(Y,\sigma)\to(Z,\vartheta)$ is $g^{**}\beta$-*continuous*, then $gof:(X,\tau)\to(Z,\vartheta)$ is $g^{**}\beta$-*continuous*.

(3) If a function $f:(X,\tau)\to(Y,\sigma)$ is $g^{**}\beta$-*continuous* and a function $g:(Y,\sigma)\to(Z,\vartheta)$ is continuous, then $gof:(X,\tau)\to(Z,\vartheta)$ is $g^{**}\beta$-*continuous*.

(4) Let (X,τ) and (Y,σ) be topological spaces. If $f:(X,\tau)\to(Y,\sigma)$ is a function, and one of the following

(a) $f^{-1}\left[Int(B)\right]\subseteq g^{**}\beta\text{-}Int\left[f^{-1}(B)\right]$ for each $B\subseteq Y$.

(b) $g^{**}\beta\text{-}Cl\left[f^{-1}(B)\right]\subseteq f^{-1}\left[Cl(B)\right]$ for each $B\subseteq Y$.

(c) $f\left[g^{**}\beta\text{-}Cl(A)\right]\subseteq Cl\left[f(A)\right]$ for each $A\subseteq X$ holds, then f is continuous.

Lemma 3.5. Let $A\subseteq Y\subseteq X$, Y is $g^{**}\beta$-*open* in X and A is $g^{**}\beta$-*open* in Y. Then A is $g^{**}\beta$-*open* in X.

Proof. Since A is $g^{**}\beta$-*open* in Y, there exists a $g^{**}\beta$-*open* set $U \subseteq X$ such that $A = Y \cap U$. Thus A being the intersection of two $g^{**}\beta$-*open* sets in X, is $g^{**}\beta$-*open* in X.

Theorem 3.6. Let $f:(X,\tau)\to(Y,\sigma)$ be a mapping and $\{U_i : i\in I\}$ be a cover of X such that $U_i \in g^{**}\beta\text{-}O(X,\tau)$ for each $i\in I$. Then prove that f is $g^{**}\beta$-*continuous*.

Proof. Let $V\subseteq Y$ be an open set, then $\left(f|U_i\right)^{-1}(V)$ is $g^{**}\beta$-*open* in U_i for each $i\in I$. Since U_i is $g^{**}\beta$-*open* in X for each $i\in I$. So by Lemma 3.5, $\left(f|U_i\right)^{-1}(V)$ is $g^{**}\beta$-*open* in X for each $i\in I$. But, $f^{-1}(V)=\cup\left\{\left(f|U_i\right)^{-1}(V) : i\in I\right\}$, then $f^{-1}(V)\in g^{**}\beta\text{-}O(X,\tau)$ because $g^{**}\beta\text{-}O(X,\tau)$ is closed under union. This implies that f is $g^{**}\beta$-*continuous*.

4. $g^{**}\beta$-IRRESOLUTE FUNCTIONS

In this section, the functions to be considered are those for which inverses of $g^{**}\beta$-*open* sets are $g^{**}\beta$-*open*. We investigate some properties and characterizations of such functions.

Definition 4.1. Let (X,τ) and (Y,σ) be topological spaces. A function $f:(X,\tau)\to(Y,\sigma)$ is called $g^{**}\beta$-*irresolute* if the inverse image $f^{-1}(U)$ of each $g^{**}\beta$-*open* set U of Y is a $g^{**}\beta$-*open* set in X.

Theorem 4.2. Let $f:(X,\tau)\to(Y,\sigma)$ be a function between topological spaces. Then the following statements are equivalent:

(1) f is $g^{**}\beta$-*irresolute;*
(2) The inverse image of each $g^{**}\beta$-*closed* set in (Y,σ) is a $g^{**}\beta$-*closed* set in (X,τ),
(3) $g^{**}\beta\text{-}Cl\left[f^{-1}(B)\right]\subseteq f^{-1}\left[g^{**}\beta\text{-}Cl(B)\right]\subseteq f^{-1}\left[Cl(B)\right]$, for each $B\subseteq Y$,
(4) $f\left[g^{**}\beta\text{-}Cl(A)\right]\subseteq g^{**}\beta\text{-}Cl\left[f(A)\right]\subseteq Cl\left[f(A)\right]$, for each $A\subseteq X$,
(5) $f^{-1}\left[g^{**}\beta\text{-}Int(B)\right]\subseteq g^{**}\beta\text{-}Int\left[f^{-1}(B)\right]$, for each $B\subseteq Y$,
(6) $g^{**}\beta\text{-}Bd\left[f^{-1}(B)\right]\subseteq f^{-1}\left[g^{**}\beta\text{-}Bd(B)\right]$, for each $B\subseteq Y$,
(7) $g^{**}\beta\text{-}b\left[f^{-1}(B)\right]\subseteq f^{-1}\left[g^{**}\beta\text{-}b(B)\right]$, for each $B\subseteq Y$,
(8) $f\left[g^{**}\beta\text{-}b(A)\right]\subseteq g^{**}\beta\text{-}b\left[f(A)\right]$, for each $A\subseteq X$,
(9) $f\left[g^{**}\beta\text{-}Cl(A)\right]\subseteq g^{**}\beta\text{-}Cl\left[f(A)\right]$, for each $A\subseteq X$.

Proof. $\mathbf{(1)\Rightarrow(2):}$ Obvious.
$\mathbf{(2)\Rightarrow(3):}$ Let $B\subseteq Y$ and $B\subseteq g^{**}\beta\text{-}Cl(B)\subseteq Cl(B)$. Then by (2) $g^{**}\beta\text{-}Cl\left[f^{-1}(B)\right]\subseteq g^{**}\beta\text{-}Cl\left[f^{-1}\left(g^{**}\beta\text{-}Cl(B)\right)\right]= f^{-1}\left[g^{**}\beta\text{-}Cl(B)\right]\subseteq f^{-1}\left[Cl(B)\right]$.
$\mathbf{(3)\Rightarrow(4):}$ Immediately by replacing B by $f(A)$ in (3).
$\mathbf{(4)\Rightarrow(1):}$ Let $W\in g^{**}\beta\text{-}O(Y)$ and $F = Y - W\in g^{**}\beta\text{-}C(Y)$. Then by (4), $f\left[g^{**}\beta\text{-}Cl\left(f^{-1}(F)\right)\right]\subseteq g^{**}\beta\text{-}Cl\left[f\left(f^{-1}(F)\right)\right]\subseteq g^{**}\beta\text{-}Cl(F)=F$. So $g^{**}\beta\text{-}Cl\left[f^{-1}(F)\right]\subseteq f^{-1}(F)$ and hence, $f^{-1}(F)=X-f^{-1}(F)\in g^{**}\beta\text{-}C(X)$, thus $f^{-1}(W)\in g^{**}\beta\text{-}O(X)$. Therefore f is $g^{**}\beta$-*irresolute.*

$(1)\Rightarrow(5)$: Let $B\subseteq Y$. Then $g^{**}\beta\text{-}Int(B)$ is $g^{**}\beta$-*open* in Y. By (1), $f^{-1}[g^{**}\beta\text{-}Int(B)]$ is $g^{**}\beta$-*open* in X. Hence $f^{-1}[g^{**}\beta\text{-}Int(B)]=g^{**}\beta\text{-}Int[f^{-1}(g^{**}\beta\text{-}Int(B))]\subseteq g^{**}\beta\text{-}Int[f^{-1}(B)]$.

$(5)\Rightarrow(6)$: Let $B\subseteq Y$. Then by (5), $f^{-1}[g^{**}\beta\text{-}Int(B)]\subseteq g^{**}\beta\text{-}Int[f^{-1}(B)]$ we have $f^{-1}(B)-g^{**}\beta\text{-}Int[f^{-1}(B)]\subseteq f^{-1}(B)-f^{-1}[g^{**}\beta\text{-}Int(B)]$. Therefore we obtain $g^{**}\beta\text{-}Bd[f^{-1}(B)]\subseteq f^{-1}[g^{**}\beta\text{-}Bd(B)]$.

$(6)\Rightarrow(5)$: Let $B\subseteq Y$. Then by (6), $g^{**}\beta\text{-}Bd[f^{-1}(B)]=f^{-1}(B)-g^{**}\beta\text{-}Int[f^{-1}(B)]\subseteq$ $f^{-1}[g^{**}\beta\text{-}Bd(B)]=f^{-1}[B-(g^{**}\beta\text{-}Int(B))]=f^{-1}(B)-f^{-1}[Int(B)]$. This implies $f^{-1}[g^{**}\beta\text{-}Int(B)]\subseteq g^{**}\beta\text{-}Int[f^{-1}(B)]$.

$(5)\Rightarrow(1)$: Let $B\subseteq Y$ be $g^{**}\beta$-*open*. Then $B=g^{**}\beta\text{-}Int(B)$. Hence by (5) we have $f^{-1}(B)=f^{-1}[g^{**}\beta\text{-}Int(B)]\subseteq g^{**}\beta\text{-}Int[f^{-1}(B)]$. Thus $f^{-1}(B)$ is $g^{**}\beta$-*open* in X. So, f is $g^{**}\beta$-*irresolute.*

$(1)\Rightarrow(7)$: Let $B\subseteq Y$, by (3), we have $g^{**}\beta\text{-}b[f^{-1}(B)]=g^{**}\beta\text{-}Cl[f^{-1}(B)]-g^{**}\beta\text{-}Int[f^{-1}(B)]\subseteq$ $f^{-1}[g^{**}\beta\text{-}Cl(B)]-g^{**}\beta\text{-}Int[f^{-1}(B)]\subseteq f^{-1}[g^{**}\beta\text{-}b(B)\cup zs\text{-}Int(B)]-g^{**}\beta\text{-}Int[f^{-1}(B)]\subseteq$ $f^{-1}[g^{**}\beta\text{-}b(B)\cup g^{**}\beta\text{-}Int(B)]-g^{**}\beta\text{-}Int[f^{-1}(g^{**}\beta\text{-}Int(B))]$. By (1) we have $g^{**}\beta\text{-}b[f^{-1}(B)]$ $\subseteq[f^{-1}(g^{**}\beta\text{-}b(B))\cup f^{-1}(g^{**}\beta\text{-}Int(B))]-f^{-1}[g^{**}\beta\text{-}Int(B)]=f^{-1}[g^{**}\beta\text{-}b(B)]$.

$(7)\Rightarrow(1)$: Let $B\in g^{**}\beta\text{-}C(Y)$ and $g^{**}\beta\text{-}b[f^{-1}(B)]\subseteq f^{-1}[g^{**}\beta\text{-}b(B)]$. Then, $g^{**}\beta\text{-}b[f^{-1}(B)]\subseteq f^{-1}[g^{**}\beta\text{-}Cl(B)-g^{**}\beta\text{-}Int(B)]=f^{-1}(B)-g^{**}\beta\text{-}Int(B)=$

$f^{-1}[g^{**}\beta\text{-}Bd(B)]\subseteq f^{-1}(B)$, we have, $f^{-1}(B)\in g^{**}\beta\text{-}C(X)$. Therefore, f is $g^{**}\beta$-*irresolute.*

$(7)\Rightarrow(8)$: Follows by replacing $f(A)$ instead of B in (7).

$(8)\Rightarrow(7)$: Let $B\subseteq Y$, by (8), we have $f[g^{**}\beta\text{-}b(f^{-1}(B))]\subseteq g^{**}\beta\text{-}b[f(f^{-1}(B))]\subseteq g^{**}\beta\text{-}b(B)$ and therefore $g^{**}\beta\text{-}b[f^{-1}(B)]\subseteq f^{-1}[g^{**}\beta\text{-}b(B)]$.

$(1)\Rightarrow(9)$: Let $A\subseteq X$. Then by (4), $f[g^{**}\beta\text{-}d(A)]\subseteq f[g^{**}\beta\text{-}Cl(A)]\subseteq g^{**}\beta\text{-}Cl[f(A)]$.

$(9)\Rightarrow(1)$: Let F be a $g^{**}\beta$-*closed* set in Y, by (7), $f[g^{**}\beta\text{-}d(f^{-1}(F))]\subseteq g^{**}\beta\text{-}Cl[f(f^{-1}(F))]\subseteq g^{**}\beta\text{-}Cl(F)=F$, then $g^{**}\beta\text{-}d[f^{-1}(F)]\subseteq f^{-1}(F)$. We have $f^{-1}(F)\in g^{**}\beta\text{-}C(X)$. Therefore f is $g^{**}\beta$-*irresolute.*

Theorem 4.3. Prove that a function $f:(X,\tau)\to(Y,\sigma)$ is $g^{**}\beta$-*irresolute* if and only if for each point p in X and each $g^{**}\beta$-*open* set B in Y with $f(p)\in B$, there is a $g^{**}\beta$-*open* set A in X such that $p\in A$, $f(A)\subseteq B$.

Proof. Necessity. Let $p\in X$ and $B\in g^{**}\beta\text{-}O(Y,\sigma)$ such that $f(p)\in B$. Let $A=f^{-1}(B)$. Since f is $g^{**}\beta$-*irresolute,* A is $g^{**}\beta$-*open* in X. Also $p\in f^{-1}(B)=A$ as $f(p)\in B$. Thus we have $f(A)=f[f^{-1}(B)]\subseteq B$.

Sufficiency. Let $B\in g^{**}\beta\text{-}O(Y,\sigma)$, and $A=f^{-1}(B)$. We show that A is $g^{**}\beta$-*open* in X. For this let $x\in A$. It implies that $f(x)\in B$. Then by hypothesis, there exists $A_x\in g^{**}\beta\text{-}O(X,\tau)$ such that $x\in A_x$ and

$f(A_x) \subseteq B$. Then $A_x \subseteq f^{-1}[f(A_x)] \subseteq f^{-1}(B) = A$. Thus $A = \cup\{A_x : x \in A\}$. It follows that A is $g^{**}\beta$-*open* in X. Hence f is $g^{**}\beta$-*irresolute.*

Definition 4.4. Let (X,τ) be a topological space. Let $x \in X$ and $N \subseteq X$. We say that N is a $g^{**}\beta$-*neighborhood* of x if there exists a $g^{**}\beta$-*open* set M of X such that $x \in M \subseteq N$.

Theorem 4.5. Prove that a function $f:(X,\tau) \to (Y,\sigma)$ is $g^{**}\beta$-*irresolute* if and only if for each x in X, the inverse image of every $g^{**}\beta$-*neighborhood* of $f(x)$, is a $g^{**}\beta$-*neighborhood* of x.

Proof. Necessity. Let $x \in X$ and let B be a $g^{**}\beta$-*neighborhood* of $f(x)$. Then there exists $U \in g^{**}\beta\text{-}O(Y,\sigma)$ such that $f(x) \in U \subseteq B$. This implies that $x \in f^{-1}(U) \subseteq f^{-1}(B)$. Since f is $g^{**}\beta$-*irresolute,* so $f^{-1}(U) \in g^{**}\beta\text{-}O(X,\tau)$. Hence $f^{-1}(B)$ is a $g^{**}\beta$-*neighborhood* of x.

Sufficiency. Let $B \in g^{**}\beta\text{-}O(Y,\sigma)$. Put $A = f^{-1}(B)$. Let $x \in A$. Then $f(x) \in B$. But then, B being $g^{**}\beta$-*open* set, is a $g^{**}\beta$-*neighborhood* of $f(x)$. So by hypothesis, $A = f^{-1}(B)$ is a $g^{**}\beta$-*neighborhood* of x. Hence by definition, there exists $A_x \in g^{**}\beta\text{-}O(X,\tau)$ such that $x \in A_x \subseteq A$. Thus $A = \cup\{A_x : x \in A\}$. It follows that A is a $g^{**}\beta$-*open* set in X. Therefore f is $g^{**}\beta$-*irresolute.*

Theorem 4.6. Prove that a function $f:(X,\tau) \to (Y,\sigma)$ is $g^{**}\beta$-*irresolute* if and only if for each x in X and each $g^{**}\beta$-*neighborhood* U of $f(x)$, there is a $g^{**}\beta$-*neighborhood* V of x such that $f(V) \subseteq U$.

Proof. Necessity. Let $x \in X$ and let U be a $g^{**}\beta$-*neighborhood* of $f(x)$. Then there exists $O_{f(x)} \in g^{**}\beta\text{-}O(Y,\sigma)$ such that $f(x) \in O_{f(x)} \subseteq U$. It follows that $x \in f^{-1}[O_{f(x)}] \subseteq f^{-1}(U)$. By hypothesis, $f^{-1}[O_{f(x)}] \in g^{**}\beta\text{-}O(X,\tau)$. Let $V = f^{-1}(U)$. Then it follows that V is a $g^{**}\beta$-*neighborhood* of x and $f(V) = f[f^{-1}(U)] \subseteq U$.

Sufficiency. Let $B \in g^{**}\beta\text{-}O(Y,\sigma)$. Put $O = f^{-1}(B)$. Let $x \in O$. Then $f(x) \in B$. Thus B is a $g^{**}\beta$-*neighborhood* of $f(x)$. So by hypothesis, there exists a $g^{**}\beta$-*neighborhood* V_x of x such that $f(V_x) \subseteq B$. Thus it follows that $x \in V_x \subseteq f^{-1}[f(V_x)] \subseteq f^{-1}(B) = O$. Since V_x is a $g^{**}\beta$-*neighborhood* of x, so there exists an $O_x \in g^{**}\beta\text{-}O(X,\tau)$ such that $x \in O_x \subseteq V_x$. Hence $x \in O_x \subseteq O$, $O_x \in g^{**}\beta\text{-}O(X,\tau)$. Thus $O = \cup\{O_x : x \in O\}$. It follows that O is $g^{**}\beta$-*open* in X. Therefore, f is $g^{**}\beta$-*irresolute.*

Theorem 4.7. Prove that a function $f:(X,\tau) \to (Y,\sigma)$ is $g^{**}\beta$-*irresolute* if and only if $f[g^{**}\beta\text{-}D(A)] \subseteq f(A) \cup [g^{**}\beta\text{-}D(f(A))]$, for all $A \subseteq X$.

Proof. Necessity. Let $f:(X,\tau) \to (Y,\sigma)$ be $g^{**}\beta$-*irresolute.* Let $A \subseteq X$, and $a_0 \in g^{**}\beta\text{-}D(A)$. Assume that $f(a_0) \notin f(A)$ and let V denote a $g^{**}\beta$-*neighborhood* of $f(a_0)$. Since f is $g^{**}\beta$-*irresolute,* so by Theorem 4.6, there exists a $g^{**}\beta$-*neighborhood* U of a_0 such that $f(U) \subseteq V$. From $a_0 \in g^{**}\beta\text{-}D(A)$, it follows that $U \cap A \neq \phi$; there exists, therefore, at least one element

$a \in U \cap A$ such that $f(a) \in f(A)$ and $f(a) \in f(V)$. Since $f(a_0) \notin f(A)$, we have $f(a) \neq f(a_0)$. Thus every $g^{**}\beta$-*neighborhood* of $f(a_0)$ contains an element of $f(A)$ different from $f(a_0)$, consequently, $f(a_0) \in g^{**}\beta\text{-}D[f(A)]$. This proves necessity of the condition.

Sufficiency. Assume that f is not $g^{**}\beta$-*irresolute.* Then by Theorem 4.6, there exists $a_0 \in X$ and a $g^{**}\beta$-*neighborhood* V of $f(a_0)$ such that every $g^{**}\beta$-*neighborhood* U of a_0 contains at least one element $a \in U$ for which $f(a) \notin V$. Put $A = \{a \in X : f(a) \notin V\}$. Then $a_0 \notin A$ since $f(a_0) \in V$, and therefore $f(a_0) \notin A$; also $f(a_0) \notin g^{**}\beta\text{-}D[f(A)]$ since $V \cap (V - \{f(a_0)\}) = \phi$. Therefore, $f(a_0) \in f[g^{**}\beta\text{-}D(A)] - [f(A) \cup (g^{**}\beta\text{-}D(f(A)))] \neq \phi$, which is a contradiction to the given condition. The condition of the Theorem is therefore sufficient, and the theorem is proved.

Theorem 4.8. Let $f:(X,\tau) \to (Y,\sigma)$ be a one-to-one function. Then f is $g^{**}\beta$-*irresolute* if and only if $f[g^{**}\beta\text{-}D(A)] \subseteq g^{**}\beta\text{-}D[f(A)]$, for all $A \subseteq X$.

Proof. Necessity. Let f be $g^{**}\beta$-*irresolute.* Let $A \subseteq X$, $a_0 \in g^{**}\beta\text{-}D(A)$ and V be a $g^{**}\beta$-*neighborhood* of $f(a_0)$. Since f is $g^{**}\beta$-*irresolute,* so by Theorem 4.6, there exists a $g^{**}\beta$-*neighborhood* U of a_0 such that $f(U) \subseteq V$. But $a_0 \in g^{**}\beta\text{-}D(A)$; hence there exists an element $a \in U \cap A$ such that $a \neq a_0$; then $f(a) \in f(A)$ and since f is one to one, $f(a) \neq f(a_0)$. Thus every $g^{**}\beta$-*neighborhood* V of $f(a_0)$ contains an element of $f(A)$ different from $f(a_0)$; consequently $f(a_0) \in g^{**}\beta\text{-}D[f(A)]$. We have therefore $f[g^{**}\beta\text{-}D(A)] \subseteq g^{**}\beta\text{-}D[f(A)]$.

Sufficiency. Follows from Theorem 4.7.

5. $g^{**}\beta$-OPEN FUNCTIONS

The purpose of this section is to investigate some characterizations of $g^{**}\beta$-*open* mappings.

Definition 5.1. Let (X,τ) and (Y,σ) be topological spaces. A function $f:(X,\tau) \to (Y,\sigma)$ is called $g^{**}\beta$-*open* if for every open set G in X, $f(G)$ is a $g^{**}\beta$-*open* set in Y.

Theorem 5.2. Prove that a mapping $f:(X,\tau) \to (Y,\sigma)$ is $g^{**}\beta$-*open* if and only if for each $x \in X$, and $U \in \tau$ such that $x \in U$, there exists a $g^{**}\beta$-*open* set $W \subseteq Y$ containing $f(x)$ such that $W \subseteq f(U)$.

Proof. Follows immediately from Definition 5.1.

Theorem 5.3. Let $f:(X,\tau) \to (Y,\sigma)$ be $g^{**}\beta$-*open.* If $W \subseteq Y$ and $F \subseteq X$ is a closed set containing $f^{-1}(W)$, then there exists a $g^{**}\beta$-*closed* $H \subseteq Y$ containing W such that $f^{-1}(H) \subseteq F$.

Proof. Let $H = Y - f(Y - F)$. Since $f^{-1}(W) \subseteq F$, we have $f^{-1}(Y-F) \subseteq (Y - W)$. Since f is $g^{**}\beta$-*open,* then H is $g^{**}\beta$-*closed* and $f^{-1}(H) = X - f^{-1}[f(X-F)] \subseteq X - (X-F) = F$.

Theorem 5.4. Let $f:(X,\tau)\to(Y,\sigma)$ be a $g^{**}\beta$-*open* function and let $B\subseteq Y$. Then $f^{-1}\left[g^{**}\beta\text{-}Cl\left(g^{**}\beta\text{-}Int\left(g^{**}\beta\text{-}Cl(B)\right)\right)\right]\subseteq Cl\left[f^{-1}(B)\right]$.

Proof. $Cl\left[f^{-1}(B)\right]$ is closed in X containing $f^{-1}(B)$. By Theorem 5.3, there exists a $g^{**}\beta$-*closed* set $B\subseteq H\subseteq Y$ such that $f^{-1}(H)\subseteq Cl\left[f^{-1}(B)\right]$. Thus, $f^{-1}\left[g^{**}\beta\text{-}Cl\left(g^{**}\beta\text{-}Int\left(g^{**}\beta\text{-}Cl(B)\right)\right)\right]\subseteq f^{-1}\left[g^{**}\beta\text{-}Cl\left(g^{**}\beta\text{-}Int\left(g^{**}\beta\text{-}Cl(H)\right)\right)\right]\subseteq f^{-1}[H]\subseteq$ $Cl\left[f^{-1}(B)\right]$.

Theorem 5.5. Prove that a function $f:(X,\tau)\to(Y,\sigma)$ is $g^{**}\beta$-*open* if and only if $f\left[Int(A)\right]\subseteq g^{**}\beta\text{-}Int\left[f(A)\right]$, *for all* $A\subseteq X$.

Proof. Necessity. Let $A\subseteq X$ and $x\in Int(A)$. Then there exists $U_x\in\tau$ such that $x\in U_x\subseteq A$. So $f(x)\in f(U_x)\subseteq f(A)$. and by hypothesis, $f(U_x)\in g^{**}\beta\text{-}O(Y,\sigma)$. Hence $f(x)\in g^{**}\beta\text{-}Int\left[f(A)\right]$. Thus $f\left[Int(A)\right]\subseteq g^{**}\beta\text{-}Int\left[f(A)\right]$.

Sufficiency. Let $U\in\tau$. Then by hypothesis, $f\left[Int(U)\right]\subseteq g^{**}\beta\text{-}Int\left[f(U)\right]$. Since $Int(U)=U$ as U is open. Also, $g^{**}\beta\text{-}Int\left[f(U)\right]\subseteq f(U)$. Hence $f(U)=g^{**}\beta\text{-}Int\left[f(U)\right]$. Thus $f(U)$ is $g^{**}\beta$-*open* open in Y. So f is $g^{**}\beta$-*open.*

Remark 5.6. The equality may not hold in the preceding Theorem.

Theorem 5.7. Prove that a function $f:(X,\tau)\to(Y,\sigma)$ is $g^{**}\beta$-*open* if and only if $Int\left[f^{-1}(B)\right]\subseteq f^{-1}\left[g^{**}\beta\text{-}Int(B)\right]$, for all $B\subseteq Y$.

Proof. Necessity. Let $B\subseteq Y$. Since $Int\left[f^{-1}(B)\right]$ is open in X and f is $g^{**}\beta$-*open*, $f\left[Int\left(f^{-1}(B)\right)\right]$ is $g^{**}\beta$-*open* in Y. Also we have $f\left[Int\left(f^{-1}(B)\right)\right]\subseteq f\left[f^{-1}(B)\right]\subseteq B$. Hence, $f\left[Int\left(f^{-1}(B)\right)\right]\subseteq g^{**}\beta\text{-}Int(B)$. Therefore $Int\left(f^{-1}(B)\right)\subseteq f^{-1}\left[g^{**}\beta\text{-}Int(B)\right]$.

Sufficiency. Let $A\subseteq X$. Then $f(A)\subseteq Y$. Hence by hypothesis, we obtain $Int(A)\subseteq Int\left[f^{-1}\left(f(A)\right)\right]\subseteq f^{-1}\left[g^{**}\beta\text{-}Int\left(f(A)\right)\right]$. Thus $f\left[int(A)\right]\subseteq g^{**}\beta\text{-}Int\left[f(A)\right]$, for all $A\subseteq X$. Hence, by Theorem 5.5, f is $g^{**}\beta$-*open.*

Theorem 5.8. Let $f:(X,\tau)\to(Y,\sigma)$ be a mapping. Then a necessary and sufficient condition for f to be $g^{**}\beta$-*open* is that $f^{-1}\left[g^{**}\beta\text{-}Cl(B)\right]\subseteq Cl\left[f^{-1}(B)\right]$ for every subset B of Y.

Proof. Necessity. Assume f is $g^{**}\beta$-*open.* Let $B\subseteq Y$. Let $x\in f^{-1}\left[g^{**}\beta\text{-}Cl(B)\right]$. Then $f(x)\in g^{**}\beta\text{-}Cl(B)$. Let $U\in\tau$ such that $x\in U$. Since f is $g^{**}\beta$-*open*, then $f(U)$ is a $g^{**}\beta$-*open* set in Y. Therefore, $B\cap f(U)\neq\phi$. Then $U\cap f^{-1}(B)\neq\phi$. Hence $x\in Cl\left[f^{-1}(B)\right]$. We conclude that $f^{-1}\left[g^{**}\beta\text{-}Cl(B)\right]\subseteq Cl\left[f^{-1}(B)\right]$.

Sufficiency. Let $B \subseteq Y$. Then $(Y-B) \subseteq Y$. By hypothesis, $f^{-1}\left[g^{**}\beta\text{-}Cl(Y-B)\right] \subseteq Cl\left[f^{-1}(Y-B)\right]$. This implies that $X - Cl\left[f^{-1}(Y-B)\right] \subseteq X - f^{-1}\left[g^{**}\beta\text{-}Cl(Y-B)\right]$. Therefore $X - Cl\left[X - f^{-1}(B)\right] \subseteq f^{-1}\left[Y - (g^{**}\beta\text{-}Cl(Y-B))\right]$. This implies that $Int\left[f^{-1}(B)\right] \subseteq f^{-1}\left[gg^{**}\beta\text{-}Int(B)\right]$. Now form Theorem 5.7, it follows that f is $g^{**}\beta$-*open*.

6. $g^{**}\beta$-CLOSED FUNCTIONS

In this section, we introduce $g^{**}\beta$-*closed* functions and study certain properties and characterizations of these types of functions.

Definition 6.1. A mapping $f:(X,\tau) \to (Y,\sigma)$ is called $g^{**}\beta$-*closed* if the image of each closed set in X is a $g^{**}\beta$-*closed* set in Y.

Theorem 6.2. Prove that a mapping $f:(X,\tau) \to (Y,\sigma)$ is $g^{**}\beta$-*closed* if and only if $g^{**}\beta\text{-}Cl\left[f(A)\right] \subseteq f\left[Cl(A)\right]$ for each $A \subseteq X$.

Proof. Necessity. Let f be $g^{**}\beta$-*closed* and let $A \subseteq X$. Then $f(A) \subseteq f\left[Cl(A)\right]$ and $f\left[Cl(A)\right]$ is a $g^{**}\beta$-*closed* set in Y. Thus $g^{**}\beta\text{-}Cl\left[f(A)\right] \subseteq f\left[Cl(A)\right]$.

Sufficiency. Suppose that $g^{**}\beta\text{-}Cl\left[f(A)\right] \subseteq f\left[Cl(A)\right]$, for each $A \subseteq X$. Let $A \subseteq X$ be a closed set. Then $g^{**}\beta\text{-}Cl\left[f(A)\right] \subseteq f\left[Cl(A)\right] = f(A)$. This shows that $f(A)$ is a $g^{**}\beta$-*closed* set. Hence f is $g^{**}\beta$-*closed*.

Theorem 6.3. Let $f:(X,\tau) \to (Y,\sigma)$ be $g^{**}\beta$-*closed*. If $V \subseteq Y$ and $E \subseteq X$ is an open set containing $f^{-1}(V)$, then there exists a $g^{**}\beta$-*open* set $G \subseteq Y$ containing V such that $f^{-1}(G) \subseteq E$.

Proof. Let $G = Y - f(X-E)$. Since $f^{-1}(V) \subseteq E$, we have $f(X-E) \subseteq Y - V$. Since f is $g^{**}\beta$-*closed*, then G is a $g^{**}\beta$-*open* set and also $f^{-1}(G) = X - f^{-1}\left[f(X-E)\right] \subseteq X - (X-E) = E$.

Theorem 6.4. Suppose that $f:(X,\tau) \to (Y,\sigma)$ is a $g^{**}\beta$-*closed* mapping. Then $g^{**}\beta\text{-}Int\left[g^{**}\beta\text{-}Cl(f(A))\right] \subseteq f\left[Cl(A)\right]$ for every subset A of X.

Proof. Suppose f is a $g^{**}\beta$-*closed* mapping and A is an arbitrary subset of X. Then $f\left[Cl(A)\right]$ is $g^{**}\beta$-*closed* in Y. Then $g^{**}\beta\text{-}Int\left[g^{**}\beta\text{-}Cl(f(Cl(A)))\right] \subseteq f\left[Cl(A)\right]$. But also $g^{**}\beta\text{-}Int\left[g^{**}\beta\text{-}Cl(f(A))\right] \subseteq g^{**}\beta\text{-}Int\left[g^{**}\beta\text{-}Cl(f(Cl(A)))\right]$. Hence $g^{**}\beta\text{-}Int\left[g^{**}\beta\text{-}Cl(f(A))\right] \subseteq f\left[Cl(A)\right]$.

Theorem 6.5. Let $f:(X,\tau) \to (Y,\sigma)$ be a $g^{**}\beta$-*closed* function, and $B, C \subseteq Y$.

Proof. (1) If U is an open neighborhood of $f^{-1}(B)$, then there exists a $g^{**}\beta$-*open* neighborhood V of B such that $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$.

(2) If f is also onto, then if $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint open neighborhoods, so have B and C.

Proof. (1) Let $V = Y - f(X - U)$. Then $V^c = Y - V = f(U^c)$. Since f is $g^{**}\beta$-*closed*, so V is a $g^{**}\beta$-*open* set. Since $f^{-1}(B) \subseteq U$, we have $V^c = f(U^c) \subseteq f[f^{-1}(B^c)] \subseteq B^c$. Hence, $B \subseteq V$, and thus V is a $g^{**}\beta$-*open* neighborhood of B. Further $U^c \subseteq f^{-1}[f(U^c)] = f^{-1}(V^c) = [f^{-1}(V)]^c$. This proves that $f^{-1}(V) \subseteq U$.

(2) If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint open neighborhoods M and N, then by (1), we have $g^{**}\beta$-*open* neighborhoods U and V of B and C respectively such that $f^{-1}(B) \subseteq f^{-1}(U) \subseteq g^{**}\beta\text{-}Int(M)$ and $f^{-1}(C) \subseteq f^{-1}(V) \subseteq g^{**}\beta\text{-}Int(N)$. Since M and N are disjoint, so are $g^{**}\beta\text{-}Int(M)$ and $g^{**}\beta\text{-}Int(N)$, hence so $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint as well. It follows that U and V are disjoint too as f is onto.

Theorem 6.6. Prove that a surjective mapping $f:(X,\tau) \to (Y,\sigma)$ is $g^{**}\beta$-*closed*, if and only if for each subset B of Y and each open set U in X containing $f^{-1}(B)$, there exists a $g^{**}\beta$-*open* set V in Y containing B such that $f^{-1}(V) \subseteq U$.

Proof. Necessity. This follows from (1) of Theorem 6.5.

Sufficiency. Suppose F is an arbitrary closed set in X. Let y be an arbitrary point in $Y - f(F)$. Then $f^{-1}(y) \subseteq X - f^{-1}[f(F)] \subseteq (X - F)$ and $(X - F)$ is open in X. Hence by hypothesis, there exists a $g^{**}\beta$-*open* set V_y containing y such that $f^{-1}(V_y) \subseteq (X - F)$. This implies that $y \in V_y \subseteq [Y - f(F)]$. Thus $Y - f(F) = \cup\{V_y : y \in Y - f(F)\}$. Hence $Y - f(F)$, being a union of $g^{**}\beta$-*open* sets, is $g^{**}\beta$-*open*. Thus its complement $f(F)$ is $g^{**}\beta$-*closed*. This shows that f is $g^{**}\beta$-*closed*.

Theorem 6.7. Let $f:(X,\tau) \to (Y,\sigma)$ be a bijection. Then the following are equivalent:

(a) f is $g^{**}\beta$-*closed*.

(b) f is $g^{**}\beta$-*open*.

(c) f^{-1} is $g^{**}\beta$-*continuous*.

Proof. $(a) \Rightarrow (b)$: Let $U \in \tau$. Then $X - U$ is closed in X. By (a), $f(X - U)$ is $g^{**}\beta$-*closed* in Y. But $f(X - U) = f(X) - f(U) = Y - f(U)$. Thus $f(U)$ is $gg^{**}\beta$-*open* in Y. This shows that f is $g^{**}\beta$-*open*.

$(b) \Rightarrow (c)$: Let $U \subseteq X$. be an open set. Since f is $g^{**}\beta$-*open*. So $f(U) = (f^{-1})^{-1}(U)$ is $g^{**}\beta$-*open* in Y. Hence f^{-1} is $g^{**}\beta$-*continuous*.

$(c) \Rightarrow (a)$: Let A be an arbitrary closed set in X. Then $X - A$ is open in X. Since f^{-1} is $g^{**}\beta$-*continuous*, $(f^{-1})^{-1}(X - A)$ $g^{**}\beta$-*open* in Y. But $(f^{-1})^{-1}(X - A) = f(X - A) = Y - f(A)$. Thus $f(A)$ is $g^{**}\beta$-*closed* in Y. This shows that f is $g^{**}\beta$-*closed*.

Remark 6.8. A bijection $f:(X,\tau) \to (Y,\sigma)$ may be open and closed but neither $g^{**}\beta$-*open* nor $g^{**}\beta$-*closed*.

7. PRE - $g^{**}\beta$ - OPEN FUNCTIONS

The purpose of this section is to introduce and discuss certain properties and characterizations of pre-$g^{**}\beta$-$open$ functions.

Definition 7.1. Let (X,τ) and (Y,σ) be topological spaces. Then a function $f:(X,\tau)\to(Y,\sigma)$ is said to be pre-$g^{**}\beta$-$open$ if and only if for each $A\in g^{**}\beta\text{-}O(X,\tau)$, $f(A)\in g^{**}\beta\text{-}O(Y,\sigma)$.

Theorem 7.2. Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\mu)$ be any two pre-$g^{**}\beta$-$open$ functions. Then the composition function $gof:(X,\tau)\to(Z,\mu)$ is a pre-$g^{**}\beta$-$open$ function.

Proof. Let $U\in g^{**}\beta\text{-}O(X,\tau)$. Then $f(U)\in g^{**}\beta\text{-}O(Y,\sigma)$. Since f is pre-$g^{**}\beta$-$open$. But then $g(f(U))\in g^{**}\beta\text{-}O(Z,\mu)$ as g is pre-$g^{**}\beta$-$open$. Hence, gof is pre-$g^{**}\beta$-$open$.

Theorem 7.3. Prove that a mapping $f:(X,\tau)\to(Y,\sigma)$ is pre-$g^{**}\beta$-$open$ if and only if for each $x\in X$ and any $U\in g^{**}\beta\text{-}O(X,\tau)$ such that $x\in U$, there exists $V\in g^{**}\beta\text{-}O(Y,\sigma)$ such that $f(x)\in V$ and $V\subseteq f(U)$.

Proof. Routine.

Theorem 7.4. Prove that a mapping $f:(X,\tau)\to(Y,\sigma)$ is pre-$g^{**}\beta$-$open$ if and only if for each $x\in X$ and for any $g^{**}\beta$-*neighborhood* U of x in X, there exists a $g^{**}\beta$-*neighborhood* V of $f(x)$ in Y such that $V\subseteq f(U)$.

Proof. Necessity. Let $x\in X$ and let U be a $g^{**}\beta$-*neighborhood* of x. Then there exists $W\in g^{**}\beta\text{-}O(X,\tau)$ such that $x\in W\subseteq U$. Then $f(x)\in f(W)\subseteq f(U)$. But $f(W)\in g^{**}\beta\text{-}O(Y,\sigma)$ as f is pre-$g^{**}\beta$-$open$. Hence $V=f(W)$ is a $g^{**}\beta$-*neighborhood* of $f(x)$ and $V\subseteq f(U)$.

Sufficiency. Let $U\in g^{**}\beta\text{-}O(X,\tau)$ and let $x\in U$. Then U is a $g^{**}\beta$-*neighborhood* of x. So by hypothesis, there exists a $g^{**}\beta$-*neighborhood* $V_{f(x)}$ of $f(x)$ such that $f(x)\in V_{f(x)}\subseteq f(U)$. It follows at once that $f(U)$ is a $g^{**}\beta$-*neighborhood* of each of its points. Therefore $f(U)$ is $g^{**}\beta$-$open$. Hence f is pre-$g^{**}\beta$-$open$.

Theorem 7.5. Prove that a function $f:(X,\tau)\to(Y,\sigma)$ is pre-$g^{**}\beta$-$open$ if and only if $f[g^{**}\beta\text{-}Int(A)]\subseteq g^{**}\beta\text{-}Int[f(A)]$, for all $A\subseteq X$.

Proof. Necessity. Let $A\subseteq X$ and $x\in g^{**}\beta\text{-}Int(A)$. Then there exists $U_x\in g^{**}\beta\text{-}O(X,\tau)$ such that $x\in U_x\subseteq A$. So $f(x)\in f(U_x)\subseteq f(A)$ and by hypothesis, $f(U_x)\in g^{**}\beta\text{-}O(Y,\sigma)$. Hence $f(x)\in g^{**}\beta\text{-}Int[f(A)]$. Thus $f[g^{**}\beta\text{-}Int(A)]\subseteq g^{**}\beta\text{-}Int[f(A)]$.

Sufficiency. Let $U\in g^{**}\beta\text{-}O(X,\tau)$. Then by hypothesis, $f[g^{**}\beta\text{-}Int(U)]\subseteq g^{**}\beta\text{-}Int[f(U)]$. Since $g^{**}\beta\text{-}Int(U)=U$ as U is $g^{**}\beta$-$open$.

Also $g**\beta\text{-}Int\left[f(U)\right]\subseteq f(U)$. Hence $f(U)=g**\beta\text{-}Int\left[f(U)\right]$. Thus $f(U)$ is $g**\beta\text{-}open$ in Y. So f is $pre\text{-}g**\beta\text{-}open$.

We remark that the equality does not hold in Theorem 7.5 as the following example shows.

Example 7.6. Let $X=Y=\{1,2\}$. suppose X is anti-discrete and Y is discrete. Let $f=Id.$, $A=\{1\}$. Then $\phi=f\left[g**\beta\text{-}Int(A)\right]\neq g**\beta\text{-}Int\left[f(A)\right]=\{1\}$.

Theorem 7.7. Prove that a function $f:(X,\tau)\to(Y,\sigma)$ is $pre\text{-}g**\beta\text{-}open$ if and only if $g**\beta\text{-}Int\left[f^{-1}(B)\right]\subseteq f^{-1}\left[g**\beta\text{-}Int(B)\right]$, for all $B\subseteq Y$.

Proof. Necessity. Let $B\subseteq Y$. Since $g**\beta\text{-}Int\left[f^{-1}(B)\right]$ is $g**\beta\text{-}open$ in X and f is $pre\text{-}g**\beta\text{-}open$, $f\left[g**\beta\text{-}Int\left(f^{-1}(B)\right)\right]$ $g**\beta\text{-}open$ in Y. Also we have $f\left[g**\beta\text{-}Int\left(f^{-1}(B)\right)\right]\subseteq f\left[f^{-1}(B)\right]\subseteq B$. Hence, $f\left[g**\beta\text{-}Int\left(f^{-1}(B)\right)\right]\subseteq g**\beta\text{-}Int(B)$. Therefore $g**\beta\text{-}Int\left[f^{-1}(B)\right]\subseteq f^{-1}\left[g**\beta\text{-}Int(B)\right]$.

Sufficiency. Let $A\subseteq X$. Then $f(A)\subseteq Y$. Hence by hypothesis, we obtain $g**\beta\text{-}Int(A)\subseteq g**\beta\text{-}Int\left[f^{-1}\left(f(A)\right)\right]\subseteq f^{-1}\left[g**\beta\text{-}Int\left(f(A)\right)\right]$. This implies that $f\left[g**\beta\text{-}Int(A)\right]\subseteq f\left[f^{-1}\left(g**\beta\text{-}Int\left(f(A)\right)\right)\right]\subseteq g**\beta\text{-}Int\left[f(A)\right]$. Thus $f\left[g**\beta\text{-}Int(A)\right]\subseteq g**\beta\text{-}Int\left[f(A)\right]$, for all $A\subseteq X$. Hence, by Theorem 7.5, f is $pre\text{-}g**\beta\text{-}open$.

Theorem 7.8. Prove that a mapping $f:(X,\tau)\to(Y,\sigma)$ is $pre\text{-}g**\beta\text{-}open$ if and only if $f^{-1}\left[g**\beta\text{-}Cl(B)\right]\subseteq g**\beta\text{-}Cl\left[f^{-1}(B)\right]$, for every subset B of Y.

Proof. Necessity. Let $B\subseteq Y$ and let $x\in f^{-1}\left[g**\beta\text{-}Cl(B)\right]$. Then $f(x)\in g**\beta\text{-}Cl(B)$. Let $U\in g**\beta\text{-}O(X,\tau)$ such that $x\in U$. By hypothesis, $f(U)\in g**\beta\text{-}O(Y,\sigma)$ and $f(x)\in f(U)$. Thus $f(U)\cap B\neq\phi$. Hence $U\cap f^{-1}(B)\neq\phi$. Therefore, $x\in g**\beta\text{-}Cl\left[f^{-1}(B)\right]$, So we obtain $f^{-1}\left[g**\beta\text{-}Cl(B)\right]\subseteq g**\beta\text{-}Cl\left[f^{-1}(B)\right]$.

Sufficiency. Let $B\subseteq Y$. Then $(Y-B)\subseteq Y$. By hypothesis, $f^{-1}\left[g**\beta\text{-}Cl(Y-B)\right]\subseteq g**\beta\text{-}Cl\left[f^{-1}(Y-B)\right]$. This implies that $X-\left[g**\beta\text{-}Cl\left(f^{-1}(Y-B)\right)\right]\subseteq X-f^{-1}\left[g**\beta\text{-}Cl(Y-B)\right]$. Hence $X-g**\beta\text{-}Cl\left[X-f^{-1}(B)\right]\subseteq f^{-1}\left[Y-\left(g**\beta\text{-}Cl(Y-B)\right)\right]$. Then this implies that $g**\beta\text{-}Int\left[f^{-1}(B)\right]\subseteq f^{-1}\left[g**\beta\text{-}Int(B)\right]$. Now by Theorem 7.7, it follows that f is $pre\text{-}g**\beta\text{-}open$.

Theorem 7.9. Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\mu)$ be two mappings such that $gof:(X,\tau)\to(Z,\mu)$ is $g**\beta\text{-}irrsolute$. Then

(1) If g is a $pre\text{-}g**\beta\text{-}open$ injection, then f is $g**\beta\text{-}irrsolute$.

(2) If f is a $pre\text{-}g**\beta\text{-}open$ surjection, then g is $g**\beta\text{-}irrsolute$.

Proof. (1) Let $U\in g**\beta\text{-}O(Y,\sigma)$. Then $g(U)\in g**\beta\text{-}O(Z,\mu)$ since g is $pre\text{-}g**\beta\text{-}open$ Also gof is $g**\beta\text{-}irrsolute$. Therefore, we have $(gof)^{-1}\left[g(U)\right]\in g**\beta\text{-}O(X,\tau)$. Since g is an injection, so we have : $(go f)^{-1}[g(U)]=(f^{-1}o g^{-1})[g(U)]= f^{-1}[g^{-1}(g(U))]=f^{-1}(U)$. Consequently $f^{-1}(U)$ is $g^{**}\beta$-*open* in X. This proves that f is gsg-*irrsolute.*

(2) Let $V\in g^{**}\beta\text{-}O(Z,\mu)$. Then $(go f)^{-1}(V)\in g^{**}\beta\text{-}O(X,\tau)$ since gof is $g^{**}\beta$-*irrsolute.* Also f is *pre*-$g^{**}\beta$-*open,* $f\left[(go f)^{-1}(V)\right]$ is $g^{**}\beta$-*open* in Y. Since f is surjective, we note that $f\left[(go f)^{-1}(V)\right]= \left[f\text{o}(go f)^{-1}\right](V)=\left[f\text{o}(f^{-1}o g^{-1})\right](V)=\left[(f o f^{-1})o g^{-1}(V)\right]=g^{-1}(V)$. Hence g is $g^{**}\beta$-*irrsolute.*

8. PRE-$g^{**}\beta$-CLOSED FUNCTIONS

In this last section, we introduce and explore several properties and characterizations of *pre*-$g^{**}\beta$-*closed* functions.

Definition 8.1. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be *pre*-$g^{**}\beta$-*closed* if and only if the image set $f(A)$ is $g^{**}\beta$-*closed* for each $g^{**}\beta$-*closed* subset A of X.

Theorem 8.2. The composition of two *pre*-$g^{**}\beta$-*closed* mappings is a *pre*-$g^{**}\beta$-*closed* mapping.

Proof. The straightforward proof is omitted.

Theorem 8.3. Prove that a mapping $f:(X,\tau)\to(Y,\sigma)$ is *pre*-$g^{**}\beta$-*closed* if and only if $g^{**}\beta\text{-}Cl[f(A)]\subseteq f[g^{**}\beta\text{-}Cl(A)]$ for every subset A of X.

Proof. Necessity. Suppose f is a *pre*-$g^{**}\beta$-*closed* mapping and A is an arbitrary subset of X. Then $f[g^{**}\beta\text{-}Cl(A)]$ is $g^{**}\beta$-*closed* in Y. Since $f(A)\subseteq f[g^{**}\beta\text{-}Cl(A)]$, we obtain $g^{**}\beta\text{-}Cl[f(A)]\subseteq f[g^{**}\beta\text{-}Cl(A)]$.

Sufficiency. Suppose F is an arbitrary $g^{**}\beta$-*closed* set in X. By hypothesis, we obtain $f(F)\subseteq g^{**}\beta\text{-}Cl[f(F)]\subseteq f[g^{**}\beta\text{-}Cl(F)]=f(F)$. Hence $f(F)=g^{**}\beta\text{-}Cl[f(F)]$. Thus $f(F)$ is $g^{**}\beta$-*closed* in Y. It follows that f is *pre*-$g^{**}\beta$-*closed*.

Theorem 8.4. Let $f:(X,\tau)\to(Y,\sigma)$ be a *pre*-$g^{**}\beta$-*closed* function, and $B,C\subseteq Y$.

(1) If U is a $g^{**}\beta$-*open* neighborhood of $f^{-1}(B)$, then there exists a $g^{**}\beta$-*open* neighborhood V of B such that $f^{-1}(B)\subseteq f^{-1}(V)\subseteq U$.

(2) If f is also onto, then if $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint $g^{**}\beta$-*open* neighborhoods, so have B and C.

Proof. (1) Let $V=Y-f(X-U)$. Then $V^c=Y-V=f(U^c)$. Since f is *pre*-$g^{**}\beta$-*closed,* so V is $g^{**}\beta$-*open.* Since $f^{-1}(B)\subseteq U$, we have $V^c=f(U^c)\subseteq f\left[f^{-1}(B^c)\right]\subseteq B^c$. Hence, $B\subseteq V$, and thus V is a $g^{**}\beta$-*open* neighborhood of B. Further $U^c\subseteq f^{-1}\left[f(U^c)\right]=f^{-1}(V^c)=\left[f^{-1}(V)\right]^c$. This proves that $f^{-1}(V)\subseteq U$.

(2) If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint gsg-*open* neighborhoods M and N, then by (1), we have gsg-*open* neighborhoods U and V of B and C respectively such that

$f^{-1}(B) \subseteq f^{-1}(U) \subseteq g^{**}\beta\text{-}g^{**}\beta\text{-}(M)$ and $f^{-1}(C) \subseteq f^{-1}(V) \subseteq g^{**}\beta\text{-}Int(N)$. Since M and N are disjoint, so are $gg^{**}\beta\text{-}Int(M)$ and $g^{**}\beta\text{-}Int(N)$, and hence so $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint as well. It follows that U and V are disjoint too as f is onto.

Theorem 8.5. Prove that a surjective mapping $f:(X,\tau) \to (Y,\sigma)$ is $pre\text{-}g^{**}\beta\text{-}closed$ if and only if for each subset B of Y and each $g^{**}\beta\text{-}open$ set U in X containing $f^{-1}(B)$, there exists a $g^{**}\beta\text{-}open$ set V in Y containing B such that $f^{-1}(V) \subseteq U$.

Proof. Necessity. This follows from (1) of Theorem 8.4.

Sufficiency. Suppose F is an arbitrary $g^{**}\beta\text{-}closed$ set in X. Let y be an arbitrary point in $Y - f(F)$. Then $f^{-1}(y) \subseteq X - f^{-1}[f(F)] \subseteq (X - F)$ and $(X - F)$ is $g^{**}\beta\text{-}open$ in X. Hence by hypothesis, there exists a $g^{**}\beta\text{-}open$ set V_y containing y such that $f^{-1}(V_y) \subseteq (X - F)$. This implies that $y \in V_y \subseteq [Y - f(F)]$. Thus $Y - f(F) = \cup\{V_y \mid y \in Y - f(F)\}$. Hence $Y - f(F)$, being a union of $g^{**}\beta\text{-}open$ sets is $g^{**}\beta\text{-}open$. Thus its complement $f(F)$ is $g^{**}\beta\text{-}closed$. This shows that f is $g^{**}\beta\text{-}closed$.

Theorem 8.6. Let $f:(X,\tau) \to (Y,\sigma)$ be a bijection. Then the following are equivalent:
(1) f is $pre\text{-}g^{**}\beta\text{-}closed$.
(2) f is $pre\text{-}g^{**}\beta\text{-}open$.
(3) f^{-1} is $g^{**}\beta\text{-}irresolute$.

Proof. $(1) \Rightarrow (2)$: Let $U \in g^{**}\beta\text{-}O(X,\tau)$. Then $X - U$ is $g^{**}\beta\text{-}closed$ in X. By (1), $f(X - U)$ is $g^{**}\beta\text{-}closed$ in Y. But $f(X - U) = f(X) - f(U) = Y - f(U)$. Thus $f(U)$ is $g^{**}\beta\text{-}open$ in Y. This shows that f is $pre\text{-}g^{**}\beta\text{-}open$.
$(2) \Rightarrow (3)$: Let $A \subseteq X$. Since f is $pre\text{-}g^{**}\beta\text{-}open$, so by Theorem 7.8, $f^{-1}[g^{**}\beta\text{-}Cl(f(A))] \subseteq g^{**}\beta\text{-}Cl[f^{-1}(f(A))]$. It implies that $g^{**}\beta\text{-}Cl[f(A)] \subseteq f[g^{**}\beta\text{-}Cl(A)]$. Thus $g^{**}\beta\text{-}Cl\left[(f^{-1})^{-1}(A)\right] \subseteq (f^{-1})^{-1}[g^{**}\beta\text{-}Cl(A)]$, for all $A \subseteq X$. Then by Theorem 4.8, it follows that f^{-1} is $g^{**}\beta\text{-}irresolute$.
$(3) \Rightarrow (1)$: Let A be an arbitrary $g^{**}\beta\text{-}closed$ set in X. Then $X - A$ is $g^{**}\beta\text{-}open$ in X. Since f^{-1} is $g^{**}\beta\text{-}irresolute$, $(f^{-1})^{-1}(X - A)$ is $g^{**}\beta\text{-}open$ in Y. But $(f^{-1})^{-1}(X - A) = f(X - A) = Y - f(A)$. Thus $f(A)$ is $g^{**}\beta\text{-}closed$ in Y. This shows that f is $pre\text{-}g^{**}\beta\text{-}closed$.

Scientific Ethics Declaration

The author declares that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the author.

Acknowledgement

* This article was presented as an oral presentation at the International Conference on Basic Sciences and Technology (www.icbast.net) held in Antalya/Turkey on November 16-19, 2023.

* The author is highly and gratefully indebted to Prince Mohammad Bin Fahd University, Al Khobar, Saudi Arabia, for providing all necessary research facilities during the preparation of this research paper.

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Author Information

Raja Mohammad Latif
Prince Mohammad Bin Fahd University
Al Khobar, Kingdom of Saudia Arabia
Contact e-mail: *rlatif@gmail.com*

To cite this article:

Latif, R. M. (2023). G**β-continuous and G**β-Irresolute mappings in topological spaces. *The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM), 25,* 168-186.