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Indeterminate Continuously Inhomogeneous Beam under End Rotation: A Longitudinal Fracture Study

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Abstract: The present paper describes a theoretical analysis of the effect of the end rotation of a statically indeterminate beam structure on the behaviour of a longitudinal crack. The beam is made of a material that is continuously inhomogeneous along the beam thickness. Besides, the material has non-linear elastic behaviour. The left end of the beam is supported with an axial double rod, while the right end is rigidly fixed. There is no external mechanical loading on the beam. Thus, the only influence on the beam is the rotation of the beam left end. Equations for determination the curvatures and the coordinates of the neutral axes are worked out and solved together with equations for resolving the static indeterminacy. The response of the longitudinal crack to the rotation of the beam left end is studied by deriving the strain energy release rate. For this purpose the balance of the energy in the beam is analyzed. The strain energy release rate is verified by applying the integral, *J*. A parametric analysis is performed to evaluate the influence of the beam end rotation and other factors on the longitudinal fracture.

Keywords: Continuously inhomogeneous structure, Longitudinal fracture, Statically indeterminate beam, End rotation, Non-linear elastic behaviour

Introduction

The increased interest for continuously inhomogeneous structural materials shown by the research community around the globe in recent decades is due in a large measure to the intensive application of functionally graded materials in different areas of engineering (Gandra et al., 2011; Toudehdehghan et al., 2017). In fact, the functionally graded materials are continuously inhomogeneous composites with a controlled continuous distribution of material properties along one or more directions (Nikbakht et al., 2019; Nagaral et al., 2019; Radhika et al., 2020; Riov, 2018). Thus, the material properties are smooth functions of one or more coordinates (Gururaja Udupa et al., 2014; Fanani et al., 2021).

In order to meet the constantly increasing requirements of up-to-date engineering towards the quality and properties of continuously inhomogeneous (functionally graded materials), different technologies for manufacturing of that sort of materials have been developed. One of the widely used technologies consists in building-up layer-by-layer (Mahamood & Akinlabi, 2017). However, the functionally graded materials manufactured by this technology have layered structure that makes them rather vulnerable to appearance of longitudinal cracks between the layers. As a result of this, one of the factors which have significant influence on the deformability, load-carrying capacity and safety of engineering structures made of such materials is the longitudinal fracture. Thus, various analyses of longitudinal fracture in continuously inhomogeneous (functionally graded) structural members have been performed recently (Carpinteri & Pugno, 2006; Rizov, 2019; Rizov & Altenbach, 2020; Tilbrook et al., 2005). These analyses usually are concerned with structures subjected to external mechanical loading. However, displacements and/or rotations of supports in statically indeterminate structures influence the longitudinal fracture behaviour even if external mechanical loading (forces, moments, etc.) is not applied on structures. This influence is subject of a theoretical analysis in the present paper.

In particular, the present paper deals with longitudinal fracture behaviour of a statically indeterminate beam structure under rotation of its left end (the beam is supported with an axial double rod in the left end, while the right end is rigidly fixed). No external mechanical loading is applied on the beam under consideration. Thus, the beam is only under the rotation of its left end. The beam is continuously inhomogeneous along the beam thickness. Besides, the beam has non-linear elastic behaviour. The strain energy release rate for the longitudinal crack under the beam left end rotation is derived. The integral, J, is used for verification. The influence of the beam end rotation on the longitudinal fracture is evaluated.

Theoretical Model

This paper addresses the problem of longitudinal fracture in the continuously inhomogeneous non-linear elastic beam structure, S_1S_4, depicted in Fig. 1.

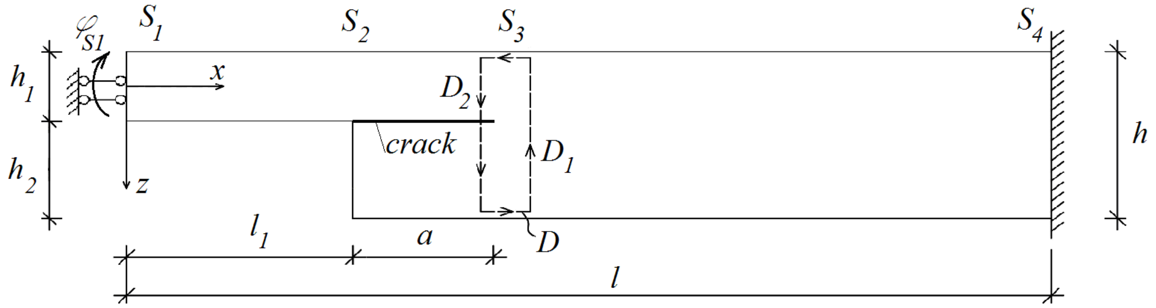

Figure1. Scheme of the beam structure.

The beam is supported with an axial double rod at its left end. The right end of the beam is rigidly fixed. Thus, the beam represents a statically indeterminate structure. The beam has three portions, S_1S_2, S_2S_3 and S_3S_4. The beam thickness in portion, S_1S_2, is h_1, while in portions, S_2S_3 and S_3S_4, the thickness is h (Fig. 1). There is a longitudinal crack in portion, S_2S_3, of the beam (the thicknesses of the upper and lower crack arms are h_1 and h_2, respectively). The left end of the beam undergoes a rotation at angle, φ_{S1}, as shown in Fig. 1. We analyze the crack response to the rotation of the beam left end. For this purpose, we derive the strain energy release rate, G, for the crack by analyzing the energy balance. This analysis yields the following expression for the strain energy release rate:

$$G = \frac{M_{S1}}{b}\frac{\partial \varphi_{S1}}{\partial a} - \frac{1}{b}\frac{\partial U}{\partial a}, \tag{1}$$

where

$$U = U_{S1S3} + U_{S3S4}. \tag{2}$$

In the above formulas, M_{S1} is the bending moment in the axial double rod at the left end of the beam, b is the beam width, a is the crack length, U is the strain energy in the beam, U_{S1S3} and U_{S3S4} are the strain energies in portions, S_1S_3 and S_3S_4, of the beam, respectively.

U_{S1S3} is determined by formula (3).

$$U_{S1S3} = (l_1 + a)\iint_{(A)} u_0 dA, \tag{3}$$

where l_1 is the length of portion, S_1S_2, of the beam, A is the area of the cross-section, u_0 is the strain energy density. The latter is defined by

$$u_0 = \int_0^{\varepsilon} \sigma d\varepsilon \,, \tag{4}$$

where σ is the stress, ε is the strain. The stress is expressed as a function of ε by the following non-linear stress-strain relation (Lukash, 1997):

$$\sigma = B\left[1 - \left(1 - \frac{\varepsilon}{\beta}\right)^n\right], \tag{5}$$

where B, n and β are material properties. The beam exhibits continuous material inhomogeneity along the thickness. The change of the material properties along the beam thickness is given by

$$B = B_{gr} + \frac{B_{dm} - B_{gr}}{h_1^{\lambda_1}}\left(\frac{h_1}{2} + z\right)^{\lambda_1}, \tag{6}$$

$$n = n_{gr} + \frac{n_{dm} - n_{gr}}{h_1^{\lambda_2}}\left(\frac{h_1}{2} + z\right)^{\lambda_2}, \tag{7}$$

$$\beta = \beta_{gr} + \frac{\beta_{dm} - \beta_{gr}}{h_1^{\lambda_3}}\left(\frac{h_1}{2} + z\right)^{\lambda_3}, \tag{8}$$

where

$$-\frac{h_1}{2} \le z \le \frac{h_1}{2}. \tag{9}$$

In formulas (6) – (9), the subscripts, gr and dm, refer to the upper and lower surface of the beam, z is the vertical centric axis of the beam, λ_1, λ_2 and λ_3 are parameters. The distribution of ε along the beam thickness is given by

$$\varepsilon = \kappa(z - z_{nn}), \tag{10}$$

where κ is the beam curvature, z_{nn} is the neutral axis coordinate.

First, we have to resolve the static indeterminacy in order to obtain the curvatures and the neutral axis coordinates in the beam portions. For this purpose, we work out the following equations:

$$\varphi_{S1} = \kappa(l_1 + a) + \kappa_1(l - l_1 - a), \tag{11}$$

$$\delta_{S1} = -\kappa z_{nn} + \kappa_1\left(\frac{h_1}{2} - \frac{h}{2} - z_{1nn}\right), \tag{12}$$

$$\iint_{(A)} \sigma dA = \iint_{(A_1)} \sigma_{S2S4} dA, \tag{13}$$

$$\iint_{(A)} \sigma z dA = \iint_{(A_1)} \sigma_{S2S4} z_1 dA, \tag{14}$$

where

$$\delta_{S1} = 0, \tag{15}$$

$$\sigma_{S3S4} = B\left[1 - \left(1 - \frac{\varepsilon_{S3S4}}{\beta}\right)^n\right], \tag{16}$$

$$\varepsilon_{S3S4} = \kappa_1 \left(z_1 - z_{1nn}\right), \tag{17}$$

$$-\frac{h}{2} \le z_1 \le \frac{h}{2}. \tag{18}$$

In formulas (11) – (17), κ_1 and z_{1nn} are the curvature and the neutral axis coordinate in portion, S_3S_4, of the beam, σ_{S3S4} is the stress, ε_{S3S4} is the strain, δ_{S1} is the horizontal displacement of the left end of the beam (δ_{S1} is zero due to the fact that the left end of the beam is constrained with an axial double rod as shown in Fig. 1), z_1 and A_1 are the vertical centric axis and the area of the cross-section of the beam in portion, S_3S_4. The curvatures and the neutral axis coordinates are determined from equations (11) – (14) by using the MatLab. ,

U_{S3S4} is determined by formula (19).

$$U_{S3S4} = \left(l - l_1 - a\right) \iint_{(A_1)} u_{01} dA, \tag{19}$$

where

$$u_{01} = \int_0^{\varepsilon_{S3S4}} \sigma_{S3S4} d\varepsilon. \tag{20}$$

The bending moment, M_{S1}, is found-out by formula (21), i.e.

$$M_{S1} = \iint_{(A)} \sigma z dA. \tag{21}$$

Then the strain energy release rate is obtained by using formula (1).

The strain energy release rate is verified by applying the integral, J, (Broek, 1986). The integration is performed along the contour, D, that has two sectors, D_1 and D_2, as shown in Fig. 1. We derive

$$J = J_{D1} + J_{D2}, \tag{22}$$

where

$$J_{D1} = \int_{D_1} \left[u_{01} \cos\alpha_{D_1} - \left(p_{x_{D_1}} \frac{\partial u}{\partial x} + p_{y_{D_1}} \frac{\partial v}{\partial x} \right) \right] ds_{D_1}, \tag{23}$$

$$J_{D2} = \int_{D_2} \left[u_0 \cos\alpha_{D_2} - \left(p_{x_{D_2}} \frac{\partial u}{\partial x} + p_{y_{D_2}} \frac{\partial v}{\partial x} \right) \right] ds_{D_2} . \quad (24)$$

The MatLab is used to perform the integration in (23) and (24). The values of the integral, J, match the strain energy release rates which is a verification of the current analysis.

Numerical Results

We apply the solution of the strain energy release rate to obtain numerical results which illustrate the influence of the beam end rotation and other factors on the longitudinal fracture behaviour of the beam under consideration.

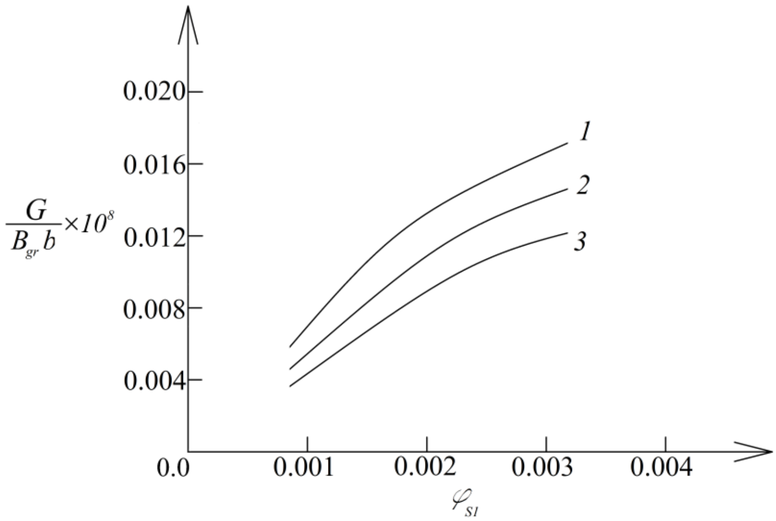

Figure 2. The strain energy release rate versus φ_{S1} (curve 1 – at $B_{dm} / B_{gr} = 0.5$, curve 2 – at $B_{dm} / B_{gr} = 1.0$ and curve 3 – at $B_{dm} / B_{gr} = 2.0$).

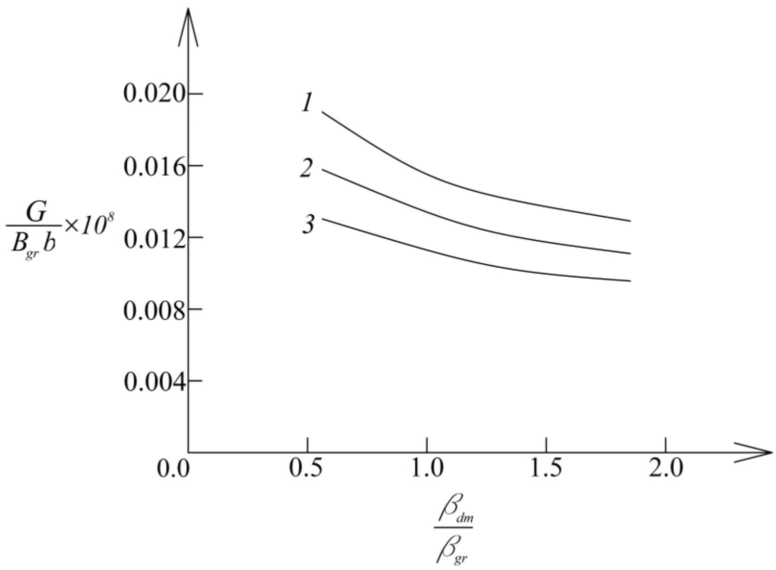

Figure 3. The strain energy release rate versus β_{dm} / β_{gr} ratio (curve 1 – at $a / l = 0.25$, curve 2 – at $a / l = 0.50$ and curve 3 – at $a / l = 0.75$).

These influences are visualized in the next four figures. The following data are used: $b = 0.010$ m, $h = 0.015$ m, $h_1 = 0.012$ m, $l = 0.400$ m, $\lambda_1 = 0.8$, $\lambda_2 = 0.8$, $\lambda_3 = 0.8$ and $\varphi_{S1} = 0.005$ rad. Figure 2 illustrates the variation of the strain energy release rate with increasing of the magnitude of the angle of rotation, φ_{S1}, at three B_{dm} / B_{gr} ratios. The curves in Fig. 2 indicate that the strain energy release rate grows at increase of φ_{S1} which is an expected behaviour.

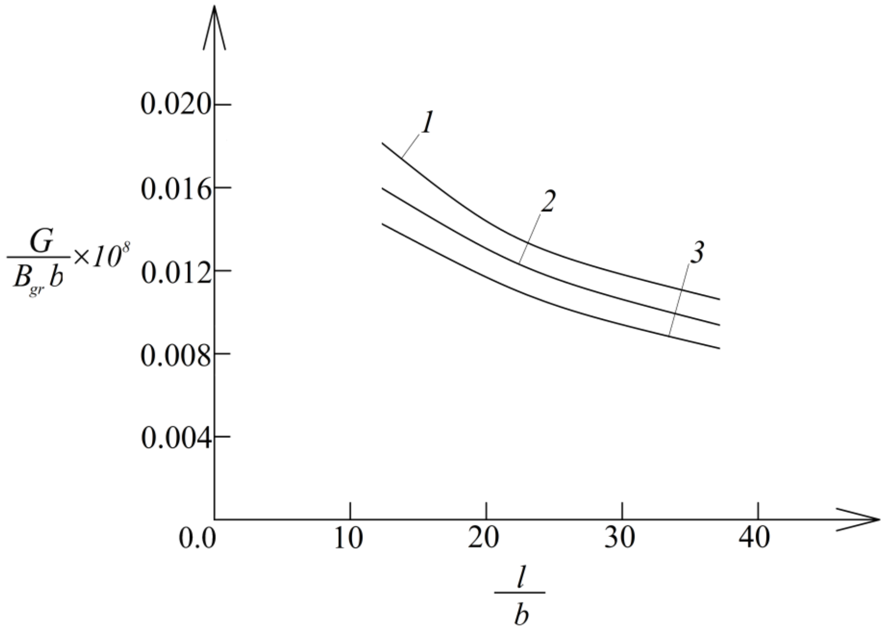

Figure 4. The strain energy release rate versus l / b ratio (curve 1 – at $n_{dm} / n_{gr} = 0.2$, curve 2 – at $n_{dm} / n_{gr} = 0.4$ and curve 3 – at $n_{dm} / n_{gr} = 0.6$).

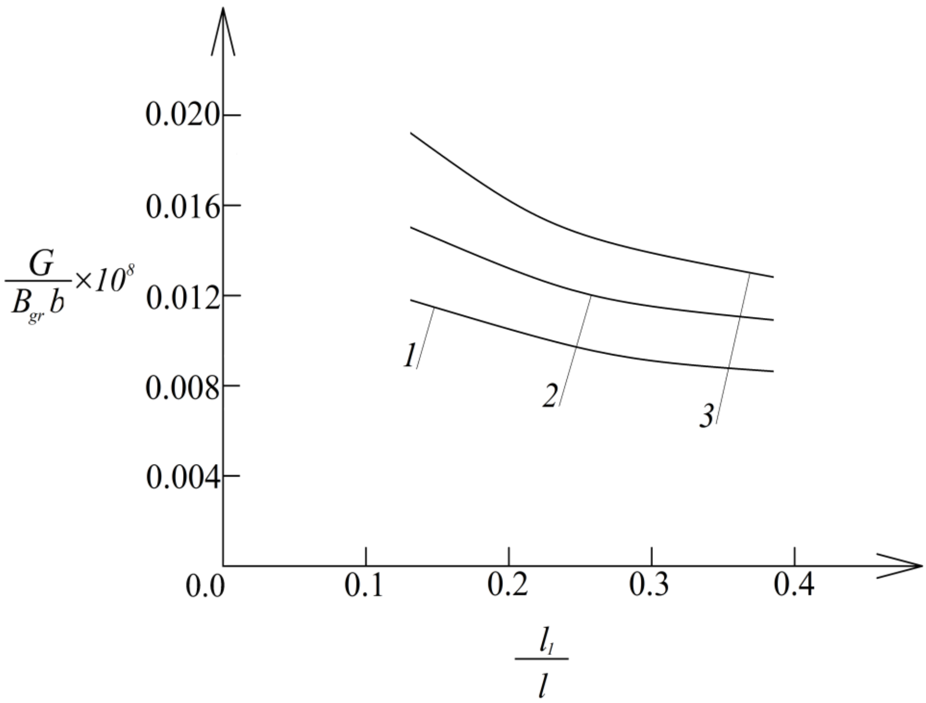

Figure 5. The strain energy release rate versus l_1 / l ratio (curve 1 – at $h / b = 1.2$, curve 2 – at $h / b = 1.2$ and curve 3 – at $h / b = 1.2$).

However, the strain energy release rate reduces when B_{dm} / B_{gr} ratio increases as one can observe in Fig. 2. The influence of β_{dm} / β_{gr} and a / l ratios on the strain energy release rate is visualized by the curves presented in Fig. 3. It can be seen in Fig. 3 that increase of β_{dm} / β_{gr} ratio causes a gradual reduction of the strain energy release rate. The strain energy release rate reduces also with increase of a / l ratio (this finding is attributed to the fact that the beam becomes more deformable).

Figure 4 shows the effect of the variation of l / b ratio on the strain energy release rate at three n_{dm} / n_{gr} ratios. The reduction of the strain energy release rate when l / b ratio grows is due again the fact that the beam deformability increases. The increase of n_{dm} / n_{gr} ratio generates a decrease of the strain energy release rate (Fig. 4). The dependence of the strain energy release rate on l / l_1 and h / b ratios is visualized by three curves in Fig. 5. The inspection of these curves reveals that increase of l / l_1 ratio generates a continuous reduction of the strain energy release rate. Increase of h / b ratios leads to growth of the strain energy release rate (Fig. 5).

Conclusion

Longitudinal fracture in a statically indeterminate beam structure under end rotation is studied theoretically. It is found that:

- the strain energy release rate grows at increase of φ_{S1};
- the strain energy release rate reduces when B_{dm} / B_{gr} ratio increases;
- increase of β_{dm} / β_{gr} ratio causes a gradual reduction of the strain energy release rate;
- the strain energy release rate reduces also with increase of a / l , l / b and l / l_1 ratio;
- increase of h / b ratio leads to growth of the strain energy release rate.

Recommendations

It is recommendable to refine the present fracture analysis by considering the viscoelastic behaviour of the statically indeterminate beam structure under end rotation.

Scientific Ethics Declaration

The author declares that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the author.

Acknowledgements or Notes

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References

Broek, D. (1986). *Elementary engineering fracture mechanics*. Springer.

Carpinteri, A., & Pugno, N. (2006). Cracks in re-entrant corners in functionally graded materials. *Engineering Fracture Mechanics, 73*(10), 1279-1291.

Fanani, E. W. A., Surojo, E., Prabowo, A. R., & Akbar, H. I. (2021). Recent progress in hybrid aluminum composite: Manufacturing and application. *Metals, 11*(12), 1919-1929.

Gandra, J., Miranda, R., Vilaca, P., Velhinho, A., & Teixeira, J.P. (2011). Functionally graded materials produced by friction stir processing. *Journal of Materials Processing Technology, 211*(11), *1659-1668.*

Udupa, G., Shrikantha Rao, S., & Rao Gangadharan, K. (2014). Functionally graded composite materials: An overview. *Procedia Materals Science, 5,* 1291-1299.

Lukash, P. (1997). *Fundamentals of non-linear structural mechanics.* Science.

Mahamood, R. C., & Akinlabi, E. T. (2017). *Functionally graded materials*. Springer.

Nagaral, M., Nayak, P. H., Srinivas, H. K., & Auradi V. (2019). Characterization and tensile fractography of Nano ZrO2 reinforced copper-zinc alloy composites. *Frattura ed Integrità Strutturale, 13*(48), 370-376.

Radhika, N., Sasikumar, J., Sylesh, J. L., & Kishore, R. (2020). Dry reciprocating wear and frictional behaviour of B4C reinforced functionally graded and homogenous aluminium matrix composites. *Journal of Materials Research and Technology*, *9*(2), 1578-1592.

Rizov, V. I. (2019). Non-linear elastic analysis of delamination in two-dimensional functionally graded multilayered beam. *International Journal of Structural Integrity, 11*(4), 1-17.

Rizov, V.I. (2019). Influence of material inhomogeneity and non-linear mechanical behaviour of the material on delamination in multilayered beams. *Frattura ed Integrità Strutturale, 13*(47), 468-481.

Rizov, V. I & Altenbach, H. (2020). Longitudinal fracture analysis of inhomogeneous beams with continuously varying sizes of the cross-section along the beam length. *Frattura ed Integrità Strutturale*, *14*(53), 38-50.

Tilbrook, M. T., Moon, R. J., & Hoffman, M. (2005). Crack propagation in graded composites. *Composite Science and Technology*, *65*(2), 201-220.

Toudehdehghan, J., Lim, W., Foo1, K. E., Ma'arof, M. I. N., & Mathews, J. (2017). A brief review of functionally graded materials. *MATEC Web of Conferences*, *131*.

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