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A SPECTRAL METHOD TO OBTAIN SOLITON SOLUTIONS TO CQNLS EQUATION

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Abstract: In optics, the cubic-quintic nonlinear Schrödinger (CQNLS) equation models electromagnetic wave propagation in various optical media. Competing cubic and quintic nonlinearities will allow the existence of stable soliton solutions (see Göksel et al. (2015)). In this study, a numerical method is introduced to obtain these solitons in different self-focusing / self-defocusing cubic-quintic media. Solitons obtained numerically by this spectral method are then validated by comparison with exact solutions.

Keywords: Nonlinear Schrödinger equation, soliton, spectral methods

Introduction

This study deals with the soliton solutions of the (1+1)D cubic-quintic nonlinear Schrödinger (CQNLS) equation. First, analytical solutions are calculated in detail for different media. Then, solutions are obtained numerically and compared with their analytical counterparts.

Analytical Solutions

Consider the following (1+1)D CQNLS equation:

$$iu_z(x,z)+u_{xx}(x,z)+\alpha\left|u(x,z)\right|^2 u(x,z)+\beta\left|u(x,z)\right|^4 u(x,z)=0 \tag{1}$$

where α and β are real constants, u corresponds to the complex-valued, slowly varying amplitude of the electric field in the x-plane propagating in the z-direction and u_{xx} corresponds to diffraction. To obtain soliton solutions, the following ansatz is used:

$$u(x,z)=f(x)e^{i\mu z} \text{ where } \lim_{x\to\pm\infty} f(x)=0 \text{ and } \mu>0\,. \tag{2}$$

Substituting

$$\begin{aligned} u_z &= i\mu f e^{i\mu z} \\ u_{xx} &= f'' e^{i\mu z} \\ |u|^2 &= f e^{i\mu z} f e^{-i\mu z} = f^2 \end{aligned} \tag{3}$$

into Eq. (1) yields

$$\left(-\mu f + f'' + \alpha f^3 + \beta f^5\right)e^{i\mu z} = 0\,. \tag{4}$$

Multiplying Eq. (4) by $2f'e^{-i\mu z}$ gives

$$2f'f'' - 2\mu f f' + 2\alpha f^3 f' + 2\beta f^5 f' = 0\,. \tag{5}$$

Integrating Eq. (5) with respect to x yields

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$$(f')^2 - mf^2 + \frac{a}{2} f^4 + \frac{b}{3} f^6 = C_1 . \tag{6}$$

The localization conditions $\lim_{x\to\pm\infty} f(x) = 0$ and $\lim_{x\to\pm\infty} f'(x) = 0$ require the integration constant C_1 to be zero:

$$(f')^2 - mf^2 + \frac{a}{2} f^4 + \frac{b}{3} f^6 = 0 . \tag{7}$$

Substituting

$$f(x) = \frac{1}{\sqrt{y(x)}} \text{ i.e. } f = y^{-0.5} \text{ and } f' = -\frac{y^{-1.5}}{2} y' \tag{8}$$

into Eq. (7) yields

$$\frac{y^{-3}}{4}(y')^2 - my^{-1} + \frac{a}{2} y^{-2} + \frac{b}{3} y^{-3} = 0 . \tag{9}$$

Multiplying Eq. (9) by $4y^3$ gives

$$(y')^2 - 4my^2 + 2ay + \frac{4b}{3} = 0 . \tag{10}$$

Eq. (10) is a separable ODE of first order as follows:

$$\frac{dy}{dx} = \pm\sqrt{4my^2 - 2ay - \frac{4b}{3}} . \tag{11}$$

Separating the variables x and y, one obtains

$$\pm 2\sqrt{m}\,dx = \frac{1}{\sqrt{y^2 - \frac{a}{2m} y - \frac{b}{3m}}} dy . \tag{12}$$

Integrating both sides of Eq. (12), i.e.

$$\pm 2\sqrt{m}\int dx = \int \frac{1}{\sqrt{y^2 - \frac{a}{2m} y - \frac{b}{3m}}} dy \tag{13}$$

results in

$$\pm 2\sqrt{m}x + \ln C = \ln\left|\sqrt{y^2 - \frac{a}{2m} y - \frac{b}{3m}} + y - \frac{a}{4m}\right| \tag{14}$$

where $\ln C$ is an integration constant. Exponentiating both sides of Eq. (14) gives

$$Ce^{\pm 2\sqrt{m}x} = \sqrt{y^2 - \frac{a}{2m} y - \frac{b}{3m}} + y - \frac{a}{4m} . \tag{15}$$

Squaring Eq. (15) yields

$$\begin{aligned} C^2 e^{\pm 4\sqrt{m}x} &= y^2 - \frac{a}{2m} y - \frac{b}{3m} + \left(y - \frac{a}{4m}\right)^2 + 2\sqrt{y^2 - \frac{a}{2m} y - \frac{b}{3m}}\left(y - \frac{a}{4m}\right) \\ &= 2y^2 - \frac{a}{m} y + \frac{a^2}{16m^2} - \frac{b}{3m} + \sqrt{y^2 - \frac{a}{2m} y - \frac{b}{3m}}\left(2y - \frac{a}{2m}\right) . \end{aligned} \tag{16}$$

Multiplying Eq. (15) by $\frac{a}{2m}$ gives

$$\frac{a}{2m} Ce^{\pm 2\sqrt{m}x} = \frac{a}{2m}\sqrt{y^2 - \frac{a}{2m} y - \frac{b}{3m}} + \frac{a}{2m} y - \frac{a^2}{8m^2} . \tag{17}$$

Adding Eq. (16) and (17) side by side, one obtains

$$C^2 e^{\pm 4\sqrt{m}x} + \frac{a}{2m} Ce^{\pm 2\sqrt{m}x} = 2y^2 - \frac{a}{2m} y - \frac{a^2}{16m^2} - \frac{b}{3m} + 2y\sqrt{y^2 - \frac{a}{2m} y - \frac{b}{3m}} . \tag{18}$$

After regrouping Eq. (18), one gets

$$C^2 e^{\pm 4\sqrt{m}x} + \frac{a}{2m} Ce^{\pm 2\sqrt{m}x} + \frac{a^2}{16m^2} + \frac{b}{3m} = 2y\left(y - \frac{a}{4m} + \sqrt{y^2 - \frac{a}{2m} y - \frac{b}{3m}}\right) \tag{19}$$

and after substituting Eq. (15) in here, one obtains

$$C^2 e^{\pm 4\sqrt{\mu}x} + \frac{\alpha}{2\mu} C e^{\pm 2\sqrt{\mu}x} + \frac{\alpha^2}{16\mu^2} + \frac{\beta}{3\mu} = 2y \times C e^{\pm 2\sqrt{\mu}x} \ . \tag{20}$$

Solving for y yields

$$y = \frac{1}{2} C e^{\pm 2\sqrt{\mu}x} + \left(\frac{\alpha^2}{32\mu^2} + \frac{\beta}{6\mu} \right) C^{-1} e^{\mp 2\sqrt{\mu}x} + \frac{\alpha}{4\mu} \ . \tag{21}$$

Substituting Eq. (21) back in Eq. (8), one obtains

$$f = \frac{1}{\sqrt{\frac{1}{2} C e^{\pm 2\sqrt{\mu}x} + \left(\frac{\alpha^2}{32\mu^2} + \frac{\beta}{6\mu} \right) C^{-1} e^{\mp 2\sqrt{\mu}x} + \frac{\alpha}{4\mu}}} \ . \tag{22}$$

The localization condition $0 = \lim_{x \to \pm\infty} f(x) = \frac{1}{\sqrt{\frac{1}{2} C e^{\pm 2\sqrt{\mu}x} + \frac{\alpha}{4\mu}}}$ requires the integration constant C to be positive:

$$C > 0 \ . \tag{23}$$

Under the condition in Eq. (23), the localization condition $0 = \lim_{x \to \mp\infty} f(x) = \frac{1}{\sqrt{\left(\frac{\alpha^2}{32\mu^2} + \frac{\beta}{6\mu} \right) C^{-1} e^{\mp 2\sqrt{\mu}x} + \frac{\alpha}{4\mu}}}$ requires

$$\alpha^2 + \frac{16}{3} \beta\mu > 0 \ , \tag{24}$$

which also implies that α and β cannot be zero at the same time:

$$(\alpha, \beta) \neq (0,0) \ . \tag{25}$$

Considering Eq. (25) and combining the conditions on μ in Eq. (2) and (24) yield

$$\begin{aligned} &0 < \mu &&, \text{ if } \beta \geq 0 \\ &0 < \mu < \frac{3\alpha^2}{16|\beta|} &&, \text{ if } \beta < 0 \end{aligned} \tag{26}$$

given that α is non-zero. If $\alpha = 0$, β and μ must be positive. For convenience, the coefficients of the exponential terms in Eq. (22) can be set equal to each other:

$$\frac{1}{2} C = \left(\frac{\alpha^2}{32\mu^2} + \frac{\beta}{6\mu} \right) C^{-1} \ . \tag{27}$$

Solving for C yields

$$C = \frac{\sqrt{\alpha^2 + \frac{16}{3}\beta\mu}}{4\mu} \ . \tag{28}$$

Note that this choice of C is compatible with Eq. (23) and (24). Substituting Eq. (28) in Eq. (22) yields

$$f = \frac{1}{\sqrt{\frac{1}{4\mu}\sqrt{\alpha^2 + \frac{16}{3}\beta\mu} \left(\frac{e^{\pm 2\sqrt{\mu}x} + e^{\mp 2\sqrt{\mu}x}}{2} \right) + \frac{\alpha}{4\mu}}} = \frac{2\sqrt{\mu}}{\sqrt{\alpha + \left(\sqrt{\alpha^2 + \frac{16}{3}\beta\mu} \right) \cosh(2\sqrt{\mu}x)}} \ . \tag{29}$$

Hence, an exact solution of Eq. (1) is

$$u(x,z) = \frac{2\sqrt{\mu}}{\sqrt{\alpha + \left(\sqrt{\alpha^2 + \frac{16}{3}\beta\mu} \right) \cosh(2\sqrt{\mu}x)}} \, e^{i\mu z} \tag{30}$$

(cf. Yang (2010)).

As it can be seen from Eq. (29), the existence of a real soliton solution depends on the values of the coefficient of the cubic nonlinearty α, the coefficient of the quintic nonlinearty β and the propagation constant μ. Is the coefficient of nonlinearty positive, then there is a so-called focusing nonlinearity. Is the coefficient of nonlinearty negative, then there is a so-called defocusing nonlinearity. The coefficients α and β may be negative, zero or positive; so there are 9 different cases to investigate (cf. Göksel (2017)). The propagation constant μ will be considered positive as set up in Eq. (2).

1) Self-defocusing cubic, self-defocusing quintic case:

In this case, α<0 and β<0. The condition in Eq. (24) becomes $\alpha^2 - \frac{16}{3}|\beta|\mu > 0$ and holds true if $\mu < \frac{3\alpha^2}{16|\beta|}$.

However, since β<0 and $\cosh(2\sqrt{\mu}x) \geq 1$, $\left(\sqrt{\alpha^2 + \frac{16}{3}\beta\mu}\right)\cosh(2\sqrt{\mu}x) < |\alpha|$ for small values of x. For instance for $x = 0$, $\alpha + \left(\sqrt{\alpha^2 + \frac{16}{3}\beta\mu}\right)\cosh(2\sqrt{\mu}x) = -|\alpha| + \sqrt{\alpha^2 + \frac{16}{3}\beta\mu} < 0$. That is, there exists no real soliton solution for positive μ values.

2) Self-defocusing cubic case:

In this case, α<0 and β=0. So, Eq. (29) becomes

$$f = \frac{2\sqrt{\mu}}{\sqrt{-|\alpha| + |\alpha|\cosh(2\sqrt{\mu}x)}} = \frac{2\sqrt{\mu}}{\sqrt{|\alpha|\left(\cosh(2\sqrt{\mu}x) - 1\right)}}. \tag{31}$$

Since α≠0 and $\cosh(2\sqrt{\mu}x) \geq 1$, f looks like a soliton except at $x = 0$ where it tends to infinity. Hence, no real soliton solution exists in this case.

3) Self-defocusing cubic, self-focusing quintic case:

In this case, α<0 and β>0. Since β>0, the condition in Eq. (24) holds true. Moreover, since β>0 and $\cosh(2\sqrt{\mu}x) \geq 1$, $\left(\sqrt{\alpha^2 + \frac{16}{3}\beta\mu}\right)\cosh(2\sqrt{\mu}x) > |\alpha|$. That is, there exist real soliton solutions for all positive μ values.

4) Self-defocusing quintic case:

In this case, α=0 and β<0. Since β<0, the condition in Eq. (24) never holds true. That is, there exists no real soliton solution for positive μ values.

5) Linear case:

In this case, α=0 and β=0. So, Eq. (7) becomes

$$(f')^2 = \mu f^2 . \tag{32}$$

After taking the square root of both sides, the following linear ODE of first order is obtained

$$f' = \pm\sqrt{\mu} f , \tag{33}$$

whose solutions are

$$f = \tilde{C}e^{\pm\sqrt{\mu}x} . \tag{34}$$

The localization condition $0 = \lim_{x\to\pm\infty} f(x) = \tilde{C}e^{\pm\sqrt{\mu}x}$ requires the integration constant $\tilde{C}$ to be zero. So, the linear case has the trivial zero solution, which is obviously not a soliton.

6) Self-focusing quintic case:

In this case, α=0 and β>0. So, Eq. (29) becomes

$$f = \sqrt{\frac{\sqrt{3}\mu}{\sqrt{\beta}\cosh(2\sqrt{\mu}x)}} . \tag{35}$$

Since β>0 and $\cosh(2\sqrt{\mu}x) \geq 1$, there exist real soliton solutions for all positive μ values.

7) Self-focusing cubic, self-defocusing quintic case:

In this case, α>0 and β<0. As in the self-defocusing cubic, self-defocusing quintic case, the condition in Eq. (24) holds true if $\mu < \frac{3\alpha^2}{16|\beta|}$. Given this and since α>0 and $\cosh(2\sqrt{\mu}x) \geq 1$, $\alpha + \left(\sqrt{\alpha^2 + \frac{16}{3}\beta\mu}\right)\cosh(2\sqrt{\mu}x) > 0$. That is, there exist real soliton solutions for $0 < \mu < \frac{3\alpha^2}{16|\beta|}$.

8) Self-focusing cubic case:
In this case, α>0 and β=0. So, Eq. (29) becomes

$$f = \frac{2\sqrt{\mu}}{\sqrt{|\alpha| + |\alpha|\cosh(2\sqrt{\mu}x)}} = \frac{2\sqrt{\mu}}{\sqrt{|\alpha|\left(\cosh(2\sqrt{\mu}x)+1\right)}}. \tag{36}$$

Since α≠0 and $\cosh(2\sqrt{\mu}x) \geq 1$, there exist real soliton solutions for all positive μ values.

9) Self-focusing cubic, self-focusing quintic case:
In this case, α>0 and β>0. Since β>0, the condition in Eq. (24) holds true. Moreover, since α>0 and $\cosh(2\sqrt{\mu}x) \geq 1$, $\alpha + \left(\sqrt{\alpha^2 + \frac{16}{3}\beta\mu}\right)\cosh(2\sqrt{\mu}x) > 0$. That is, there exist real soliton solutions for all positive μ values.
The results of these 9 cases are summarized in Figure 1.

α \ β	negative	zero	positive
negative	x	x	✓
zero	x	x	✓
positive	soliton exists for 0<μ<3α²/(16|β|)	✓	✓

Figure 1. Existence of analytical solutions of the (1+1)D CQNLS equation

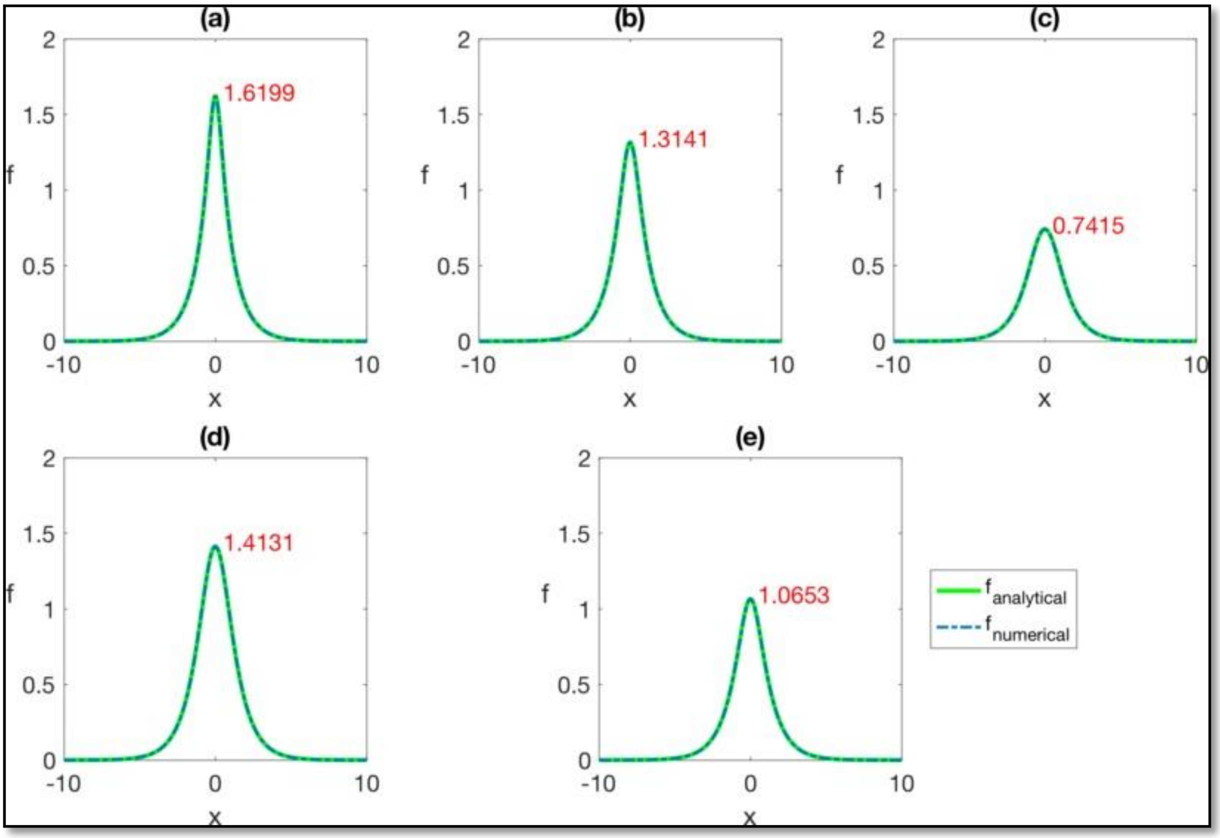

Figure 2. Numerical solutions ($f_{numerical}$) of the (1+1)D CQNLS equation in comparison with the corresponding analytical solutions ($f_{analytical}$) in different media: (a) α=-1, β=1, (b) α=0, β=1, (c) α=4, β=-1, (d) α=1, β=0, (e) α=1, β=1

Numerical Solutions

Solutions are also obtained numerically using Spectral Renormalization Method by Ablowitz et al. (2005). Figure 2 represents selected solitons in different media, namely in:
a) self-defocusing cubic, self-focusing quintic
b) self-focusing quintic
c) self-focusing cubic, self-defocusing quintic
d) self-focusing cubic
e) self-focusing cubic, self-focusing quintic
media. No soliton could be obtained for the other cases, as expected. The red numbers by the peak of solitons in Figure 2 mark their maximum amplitudes.

Conclusion

In this work, soliton solutions of the (1+1) CQNLS equation are obtained analytically and numerically for different media. It is seen that the numerical solutions are in perfect agreement with the analytical ones. This validates the numerical method and is very important for the cases where an analytical solution does not exist.

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