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Handling Growth and Decay Problems by the New General Integral Transform

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Abstract: The mathematical techniques known as integral transformations are very adaptable and may be used in a multitude of domains including applied sciences, engineering, and mathematics. Numerical calculations and theories involving differential equations have made considerable use of integral transform techniques. Complex functions can be broken down into more manageable functions that can be solved and examined via integral transformations. One of the most prevalent applications of exponential functions involves growth and decay models. Exponential growth and decay show up in a host of natural applications. Almost everything in our world grows and decays. The growth and decay problems arise in the field of chemistry, physics, biology, social science, zoology. The processes of growth and decay are expressed in terms of mathematical models. In this paper, an integral transform called as "New General Integral Transform" is employed for solving growth and decay problems and some applications are given to demonstrate the effectiveness of this transform for growth and decay problems.

Keywords: Growth problem, Decay problem, Integral transform

Introduction

One of the most prevalent applications of exponential functions involves growth and decay models. Exponential growth and decay show up in a host of natural applications. Almost everything in our world grows and decays. The lifespan of people in actuarial science, radioactive elements, virus propagation in biology, rise and fall of current in L-R circuits, water discharge in a vessel through an orifice, business investment value, melting of ice in the polar regions, sun or moon rising time, statistical breakdown of epidemics, and thermal radiation by black bodies are some of the applications of growth and decay problems.

Few concepts are as pervasive and powerful in the fabric of mathematical modeling as growth and decay. From growing populations to vanishing radioactive isotopes, the phenomena of growth and decay permeate nearly every aspect of our natural and engineered worlds. At the heart of these phenomena lies a rich tapestry of mathematical principles that offer insights into the dynamic processes that shape the ebb and flow of quantities over time.

At its core, growth and decay problems revolve around understanding how the quantity of a given asset changes over time. Whether it is the exponential growth of bacteria in a petri dish or the gradual dissipation of heat from a cooling object, the underlying dynamics follow different patterns that can be measured and analyzed using mathematical models. These models serve as powerful forecasting tools that allow us to predict future situations based on current conditions and historical trends.

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The processes of growth and decay are expressed in terms of mathematical models. The growth and decay problems are modelled mathematically by the same differential equation as

$$\frac{d\,y(t)}{dt} = k\,y(t) \tag{1}$$

with initial condition as

$$y(0) = y_0$$

where $y(t)$ is the amount of the substance at t, y_0 is the amount of the substance at $t = 0$ and k is the constant of proportionality. If $k > 0$, equation (1) is called growth problem and if $k < 0$, equation (1) is called decay problem (Kapur, 2005; Bronson & Costa, 2006; Gorain, 2014).

Method

A New General Integral Transform

Integral transforms are powerful mathematical tools used to solve linear and nonlinear problems in many fields such as applied mathematics, physics, engineering, and others. By transforming a differential equation into a different domain, such as the frequency domain or the Laplace domain, integral transforms simplify the problem-solving process and provide new insights into the dynamics of the system (Debnath & Bhatta, 2007). In this part, a new general integral transform which comprises most of the integral transform in the family of Laplace transform.

Definition 1. (Jafari, 2021) Let $f(t)$ be an integrable function defined for $t \geq 0$, $p(s)$ and $q(s)$ be positive real functions. Then, the new general integral transform $\mathcal{T}(s)$ of $f(t)$ is defined as follows,

$$T\{f(t);s\} = \mathcal{T}(s) = p(s)\int_0^{\infty} f(t)e^{-q(s)t}\,dt,$$

provided the integral exists for some $q(s)$. It is essential to note that this transform for those $f(t)$ is not continuously differentiable contains terms with negative or fractional powers of $q(s)$.

Theorem 1 (Existence Theorem). (Jafari, 2021) If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geq 0$ and is of exponential order k, that is $|f(t)| \leq M\,e^{kt}$ for some positive real number M and k, then $\mathcal{T}(s)$ exists for all $q(s) > k$.

Proof: Since $f(t)$ is piecewise continuous, $e^{-st}f(t)$ is integrable over any finite interval on the t-axis. Assume that $q(s) > k$. Then, the proof of the existence of the $\mathcal{T}(s)$ is obtained as follows:

$$\left\|\mathcal{T}(s)\right\| = \left\|T\{f(t);s\}\right\| = \left|p(s)\int_0^{\infty} f(t)e^{-q(s)t}dt\right| \leq p(s)\int_0^{\infty} Me^{kt}e^{-q(s)t}dt = \frac{M\,p(s)}{q(s)-k}.$$

In Table 1, one can find the transformation of some fundamental functions.

Table 1. The new general integral transform of some fundamental functions

f(t)	$\mathcal{T}(s)$
1	$\frac{p(s)}{q(s)}$
t	$\frac{p(s)}{\left(q(s)\right)^2}$
t^{α} , $\alpha > 0$	$\frac{\Gamma(\alpha+1)\,p(s)}{\left(q(s)\right)^{\alpha+1}}$, $\alpha > 0$
$e^{\alpha t}$	$\frac{p(s)}{q(s)-\alpha}$, $q(s) > \alpha$
$\sin(ct)$	$\frac{c\,p(s)}{c^2+\left(q(s)\right)^2}$
$\cos(ct)$	$\frac{p(s)\,q(s)}{c^2+\left(q(s)\right)^2}$

Theorem 2 (Transform of Derivatives). (Jafari, 2021) Let $f(t)$ be differentiable and $p(s)$ and $q(s)$ be positive real functions. Then,

1. $T\{f'(t);s\} = q(s)\;T\{f(t);s\} - p(s)f(0)$
2. $T\{f''(t);s\} = q^2(s)\;T\{f(t);s\} - q(s)p(s)f(0) - p(s)f'(0)$
3. $T\left\{f^{(n)}(t);s\right\} = q^n(s)\;T\{f(t);s\} - p(s)\sum_{k=0}^{n-1} q^{n-1-k}(s)f^{(k)}(0)$

Application to Growth and Decay Problems

In this section, the new general integral transform will be applied to growth and decay model.
Firstly, taking the transform of the equation (1), we obtain

$$\mathcal{T}\left[\frac{d\,y(t)}{dt}\right] = \mathcal{T}\left[k\,y(t)\right] \tag{2}$$

Now, rearranging the equation (2) according to the theorem 2 and Table 1, and substituting the value in the initial condition into this equation, we obtain

$$q(s)\mathcal{T}(s) - p(s)y_0 = k\left[\mathcal{T}(s)\right] \tag{3}$$

where $\mathcal{T}(s) = \mathcal{T}\left[y(t)\right]$.

If we make the equation (3) suitable for applying the inverse transform, we obtain

$$\mathcal{T}(s) = y_0\,\frac{p(s)}{q(s)-k} \tag{4}$$

Finally, applying the inverse transform to the equation (4) using Table 1, we find the solution as

$$y(t) = y_0\,e^{kt}\,. \tag{5}$$

We will examine the effectiveness of this method through some numerical applications and compare the results with those obtained with other methods in the literature.

Application 1. (Aggarwal et al., 2018) Any radioactive substance is known to decay in proportion to the amount available. Consider that $500\,mg$ of a radioactive substance is initially available and after five hours the radioactive substance has lost 25 percent of its available mass. Find the half-life of this radioactive substance. Using the equation (5) according to the values given in the question, we can write

$$y(t) = 500e^{kt}$$

Since 25 percent of the available mass of this radioactive substance is lost at $t = 5$, we have

$$N = 500 - 125 = 375.$$

Now, we can write

$$375 = 500e^{-5k}$$

$$e^{-5k} = 0.75$$

$$k = -\frac{1}{5}\ln(0.75) = 0.0575.$$

We are looking for t when $N = \frac{N_0}{2} = 250$,

$$250 = 500e^{-(0.0575)t}$$

$$t = 12.05$$

which is in good agreement with the results obtained by other methods (Aggarwal et al., 2018; Peker & Cuha, 2022).

Application 2. (Aggarwal et al., 2020) Any radioactive matter is known to decay in proportion to the amount available. Find the half-life of the radioactive matter for the case where $100\,mg$ of radioactive matter is initially available and after six hours the radioactive matter has lost 30 percent of its available mass. Using the equation (5) according to the values given in the question, we can write

$$y(t) = 100e^{kt}$$

Since the radioactive matter lost 30 percent of its available mass at $t = 6$, we have

$$N = 100 - 30 = 70.$$

Now, we can write

$$70 = 100e^{-6k}$$

$$e^{-6k} = 0.7$$

$$k = -\frac{1}{6}\ln(0.7) = 0.059.$$

We are looking for t when $N = \frac{N_0}{2} = 50$,

$$50 = 100e^{-(0.059)t}$$

$$t = 11.75$$

which is exactly coincides with the results obtained by other methods (Aggarwal et al., 2020; Peker & Cuha, 2022).

Application 3. (Rao, 2017) It is known that the rate of degradation of a given substance in a given solution at any instant is proportional to the amount present in the solution at that moment. It is known that there are $27\,grams$ of substance in a solution initially and $8\,grams$ of this substance remain after three hours. How much substance remains in the solution after another one hour has passed?

Using the equation (5) according to the values given in the question, we can write

$$8 = 27e^{-3k}$$

$$e^{-k} = \left(\frac{8}{27}\right)^{\frac{1}{3}}$$

In order to find remaining substance in the solution at $t = 4$, we write

$$\begin{aligned} y(4) &= 27e^{-4k} \\ &= 27\left(\frac{8}{27}\right)^{\frac{4}{3}} \\ &= \frac{16}{3}. \end{aligned}$$

This result is fairly consistent with the ones obtained by other methods (Rao, 2017; Peker & Çuha, 2022).

Conclusion

To sum up, growth and decay problems serve as a cornerstone in the structure of mathematical modeling and provide a lens through which we can unravel the dynamic interaction of quantities over time. From the rapid proliferation of populations to the gradual dissipation of energy, the principles of growth and decay permeate countless fields, shaping our understanding of natural phenomena and informing prediction and control strategies.

Utilizing the new general integral transform approach signifies a noteworthy progression in the realm of mathematical analysis and has the potential to transform the resolution of intricate differential equations across diverse scientific and engineering fields.

Recommendation

This new general integral transform can be applied easily, effectively and reliably to various models found in other applied sciences.

Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

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