The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM)
ISSN: 2602-3199

The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM), 2024

Volume 28, Pages 175-184

ICBASET 2024: International Conference on Basic Sciences, Engineering and Technology

A Fixed Point Theorem on Partial Metric Spaces of Hyperbolic Type

Silvja Cobani
Fan S. Noli University

Elida Hoxha
University of Tirana

Abstract: In this research paper, we introduce the concept of partial metric spaces of hyperbolic type. When it comes to hyperbolic spaces, they are mostly studied in the context of metric spaces. A partial metric space is a generalization of a metric space, where self-distance is not necessarily zero. This concept became particularly interesting when Kumar et al. (2017) introduced and studied convex partial metric spaces. His result were useful in defining partial metric spaces of hyperbolic type, which is the kickoff point of our paper. After this, we focus our study in providing a proof of the existence of a fixed point for a non-self-mapping of a specific contracting type that was first introduced by Ćirić (2006). Our result is a generalization of the results of Ćirić and other cited authors. In the end an example is provided. This example serves to illustrate the applicability of our fixed point theorem and shows that results from metric spaces of hyperbolic type can be extended to partial metric spaces of hyperbolic type.

Keywords: Partial metric space, Non-self-mapping, Contraction, Fixed point

Introduction

Fix point theory is a branch of mathematics that arouses interest with its various applications in different fields such as nonlinear analysis, integral and differential equations, dynamic systems, fractals etc. The Banach Contraction Principle is well known as a useful tool with wide applications. It states that if (X, d) is a complete metric space, and the self- mapping $T: X \to X$ satisfies $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$, where $0 < \lambda < 1$, then T has a unique fixed point. This classical theorem has been generalized and studied extensively. Ćirić (1974) introduced and studied self-mappings on K, a nonempty closed subset of X, which satisfies:

$$d(Tx, Ty) \leq \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \text{ where } 0 < \lambda < 1.$$

Whereas Boyd and Wong (1969) investigated mappings that satisfy the condition: $d(Tx, Ty) \leq \varphi(d(x, y))$, where $\varphi: R^+ \to R^+$, called a comparison function, is upper semi-continuous from the right and satisfies the condition $\varphi(t) < t$ for all $t > 0$. Subsequently, Ćirić (2006) extended these finding to non- self-mappings and proved some theorems related to fixed points on hyperbolic type metric spaces. Izadi demonstrated in 2012 that Ćirić's findings are also applicable to quasi-metric spaces of hyperbolic type.

Partial metric spaces, as a generalization of metric spaces, were introduced by Matthews (1992). In these type of spaces, the distance from a point to itself might not always be zero. In other words, there may be self-distances $d(x, x)$ that may not be zero. In generalizing the metric space in this way some of its properties may be lost, but Matthews proved that the well- known Banach Contracting Principle can be extended to partial metric spaces as well. Since then partial metric spaces, their properties, fixed points and their applications have been the focus of many studies, especially in the field of computer science. Refer for example to Alghamdi et al. (2013) and Han et al. (2017) and Bugajewski et al. (2022) and the references there in.

- Selection and peer-review under responsibility of the Organizing Committee of the Conference

In recent years the study of partial metric spaces has shifted to the study of convexity. In 2017, Kumar et al. studied some properties of metrically convex partial metric spaces and proved some results on fixed points in these spaces. He did so by generalizing the following definition:

Definition 1.1. (Assad & Kirk, 1972- Menger) A metric space (X,d) is *metrically convex* if X is such that for each $x,y \in X$ with $x \neq y$ there exists $z \in X, x \neq z \neq y$ such that:

$d(x,z) + d(z,y) = d(x,y)$.

If (X,d) is a metrically convex metric space and $x,y \in X$, *a metric segment* is defined by:

$seg[x,y] \coloneqq \{z \in X: d(x,z) + d(z,y) = d(x,y)\}$.

In our paper we will be referring to a type of metrically convex metric spaces, as defined by Kirk (1982).

Definition 1.2. (Kirk, 1982) A metric space (X,d) is called *a metric space of hyperbolic type* if it contains a family L of *metric segments* such that:

a) each two points $x,y \in X$ are endpoints of exactly one member $seg[x,y] \in L$, and
b) if $u,x,y \in X$ and $z \in seg[x,y]$ is such that $d(x,z) = \lambda d(x,y)$ for $\lambda \in [0,1]$, then$d(u,z) \leq (1-\lambda)d(u,x) + \lambda d(u,y)$

The purpose of our paper is to generalize this concept for partial metric spaces and show that Ćirić (2006) fixed point results can be extended to the partial metric space of hyperbolic type. Thus we demonstrate that despite the generalization certain properties of metric spaces of hyperbolic type can be preserved.

With this in mind we start our paper by recalling some useful basic definitions from partial metric spaces and by using some preliminary results from Kumar et al. (2017). In the main section of the paper we define the partial metric space of hyperbolic type and prove a fixed point theorem on this type of partial metric space. Our result is a generalization of Ćirić (2006) and Izadi (2012) results.

Preliminaries

We will start by recalling some basic definitions and properties of partial metric spaces.

Definition 2.1. (Matthews, 1992) Let X be a nonempty set. A *partial metric* is a function $p: X \times X \to R^+$ such that for all $x,y,z \in X$, the following axioms are satisfied:

p1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
p2) $p(x,x) \leq p(x,y)$,
p3) $p(x,y) = p(y,x)$,
p4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is called *partial metric space* and $p(x,x)$ is called *size of* x. A closer look on these axioms reveals that if $x = y$ then $p(x,y)$ might not be zero.

According to Matthews (1994), every partial metric p induces a metric $d_p: X \times X \to R^+$ defined by:

$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ for all $x,y \in X$.

An example of a partial metric space is (X,p) with $X = R^+$and $p(a,b) = \max\{a,b\}, \forall\, a,b \geq 0$. In this case the derived metric is $d_p(a,b) = |a-b|$. For other examples see Bukatin et al. (2009) and Matthews (1994).

Definition 2.2. (Matthews, 1994) Let (X,p) be a partial metric space and $\{x_n\}$ a sequence in X. Then

a) $\{x_n\}$ *converges* to $x \in X$ if and only if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$.
b) $\{x_n\}$ is called a *Cauchy sequence* if and only if $\lim_{n\to\infty} p(x_n,x_m)$ exists and is finite.

c) if every Cauchy sequence $\{x_n\}$ converges to $x \in X$, meaning that $\lim_{n\to\infty} p(x_n, x_m) = p(x,x)$, then (X,p) is said to be *a complete partial metric space*.

It should be noted that the limit of a sequence in partial metric space is not necessary unique.

Each partial metric p on X generates a T_0- topology $T(p)$ on X which has as a base the family of open p- balls $\{B_p(x,\varepsilon): x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X: p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 2.3. Let (X,p) be a partial metric space and A a subset of X.

1) A is called an *open set* if for $x \in A$ there exists $B_p(x,\varepsilon)$ such that $B_p(x,\varepsilon) \subset A$.
2) A is called a *closed set* if its complement is open.
3) A point $x \in A$ is called *a limit point* of A if there exists a sequence $\{x_n\} \subset A$ such that $\lim_{n\to\infty} p(x_n, x) = p(x,x)$. The set A together with all its limit points is called *closure* of A.
4) *The boundary of* A is denoted by ∂A and is the intersection of the closure of A with the closure of its complement.
5) A is called *bounded* if there exists $M > 0$ such that $p(x,y) \leq M$, for every $x, y \in A$.
6) If A is bounded then $diam(A) = \sup\{p(x,y): x,y \in A\} < +\infty$ is called *the diameter of* A.

Next we turn to the results of Kumar et al. (2017) for the definition and some useful properties of metrically convex partial metric spaces.

Definition 2.4. (Kumar et al., 2017) A partial metric space (X,p) is said to be *metrically convex* if the corresponding metric space (X, d_p) is metrically convex.

If (X,p) is a metrically convex partial metric space and $x, y \in X$, then following the results of Kumar et al. (2017) we can define *a metric segment* (isometric image of a real line segment) to be:

$seg[x,y] \coloneqq \{z \in X: p(x,y) + p(z,z) = p(x,z) + p(z,y)\}$.

As an example, we revisit the partial metric space (R^+, p), where $p(a,b) = \max\{a,b\}, \forall\, a,b \in R^+$. This space is also metrically convex because the derived metric space (R^+, d_p), where $d_p(a,b) = |a-b|, \forall\, a,b \in R^+$, is metrically convex.

Lemma 2.5. (Kumar et al., 2017) Let K be non-empty closed subset of a metrically convex partial metric space (X,p). If and $x \in K$ and $y \notin K$ then there exists $z \in \partial K$, such that $p(x,y) + p(z,z) = p(x,z) + p(z,y)$.

Definition 2.6. The function $\varphi: R^+ \to R^+$ is called an ultra- altering distance if:

1) φ is non- decreasing,
2) $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0$.

A φ-contractive condition alone does not guarantee the existence of a fixed point, unless additional conditions are assumed. Therefore, to ensure the existence of a fixed point under the contractive condition of an ultra-altering distance φ, various authors have employed the following additional conditions on φ: φ is upper semi-continuous (Boyd & Wong, 1969) ; φ is non- decreasing and continuous from the right (Park & Roades, 1981); φ is non- decreasing and $\frac{t}{t-\varphi(t)}$ is non- decreasing (Carbone et al., 1989); φ is non- decreasing and $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$ (Jachymski, 1994). In our paper we will be working with an ultra- altering distance that is lower semi- continuous and such that $\lim_{n\to\infty}(t - \varphi(t)) = +\infty$ (Ćirić, 2006).

Main Results

We introduce the partial metric space of hyperbolic type.

Definition 3.1. A partial metric space (X,p) is called *a partial metric space of hyperbolic type* if the corresponding metric space (X, d_p) is of hyperbolic type.

We note that a partial metric space of hyperbolic type is metrically convex.

Lemma 3.2. Let (X,p) be a partial metric space of hyperbolic type. If $u,x,y \in X$ and $z \in seg[x,y]$ is such that $d_p(x,z) = \lambda d_p(x,y)$ for $\lambda \in [0,1]$, then $d(u,z) \leq 2\max\{d(u,x), d(u,y)\}$.

Proof. By the definition 3.1 and 1.2, for $u,x,y \in X$ and $z \in seg[x,y]$ is such that $d_p(x,z) = \lambda d_p(x,y)$ for $\lambda \in [0,1]$, we have

$$d_p(u,z) \leq (1-\lambda)d_p(u,x) + \lambda d_p(u,y)$$

$$2p(u,z) - p(u,u) - p(z,z) \leq (1-\lambda)[2p(u,x) - p(u,u) - p(x,x)] + \lambda[2p(u,y) - p(u,u) - p(y,y)]$$

$$2p(u,z) - p(z,z) \leq (1-\lambda)[2p(u,x) - p(x,x)] + \lambda[2p(u,y) - p(y,y)].$$

Since p is e partial metric, by the Definition 2.1 we have that

$$p(u,z) - p(z,z) \geq 0 \qquad 0 \leq p(u,x) - p(x,x) \leq p(u,x) \qquad 0 \leq p(u,y) - p(y,y) \leq p(u,y).$$

Therefore,

$$p(u,z) \leq p(u,z) + [p(u,z) - p(z,z)] \leq 2[(1-\lambda)p(u,x) + \lambda p(u,y)] \leq 2\max\{p(u,x), p(u,y)\}. \blacksquare$$

Theorem 3.3. Let (X,p) be a complete partial metric space of hyperbolic type, K a nonempty closed subset of X and $T: K \to X$ a non- self mapping such that:

(i) $T(\partial K) \subseteq K$,

(ii) $p(Tx,Ty) \leq \varphi\left(\frac{1}{2}\max\{p(x,y), p(x,Tx), p(y,Ty), p(x,Ty), p(y,Tx)\}\right)$, where the function φ is an ultra-alternating distance, lower semi- continuous and such that $\lim_{n\to\infty}(t - \varphi(t)) = +\infty$.

Then T has a unique fixed point in K.

Proof. The theorem can be proved in five steps.

Step 1. We start by constructing a sequence $\{x_n\}$. First we choose $x_0 \in \partial K$. Then (i) implies that $Tx_0 \in K$ and we set $x_1 = Tx_0$. If $Tx_1 \in K$, then $x_2 = Tx_1$. If $Tx_1 \notin K$, since we have that $x_1 \in K$ and that (X,p) is of hyperbolic type, Lemma 2.5 implies that there exists $x_2 \in \partial K$ such that $x_2 \in seg[x_1, Tx_1]$, i.e. such that

$$p(x_1,Tx_1) + p(x_2,x_2) = p(x_1,x_2) + p(x_2,Tx_1).$$

Following this reasoning, we iteratively construct the sequences $\{x_n\}$ and $\{Tx_n\}$ in K such that for all $n \geq 2$,

$x_n = Tx_{n-1}$, if $Tx_{n-1} \in K$ or $x_n \in \partial K$ and $x_n \in seg[x_{n-1}, Tx_{n-1}]$, if $Tx_{n-1} \notin K$.

(i.e. $p(x_{n-1},Tx_{n-1}) + p(x_n,x_n) = p(x_{n-1},x_n) + p(x_n,Tx_{n-1})$, if $Tx_{n-1} \notin K$.)

Step 2. We show that the sequences we construsted are bounded. First, for $n \geq 1$, we define

$$A_n = \{x_i\}_{i=0}^{n-1} \cup \{Tx_i\}_{i=0}^{n-1} \text{ and } \alpha_n = diam(A_n)$$

If $\alpha_n = 0$, Definition 2.1. implies $Tx_0 = x_0$ and this proves the theorem.

If $\alpha_n > 0$, we show that $\alpha_n = p(x_0, Tx_k)$, for some $k \in \{0,1,\dots,n-1\}$. We consider the following cases.

Case 1. Let $\alpha_n = p(x_i, Tx_k)$ for some $i,k \in \{0,1,\dots,n-1\}$. To prove that $x_i = x_0$, we suppose to the contrary that $x_i \neq x_0$. Then $x_{i-1} \in \{x_n\} \subseteq K$ and Tx_{i-1} is defined.

(a) If $Tx_{i-1} \in K$, then by construction of $\{x_n\}$, $x_i = Tx_{i-1}$. Thus by condition (ii) it follows that:

$$\alpha_n = p(x_i,Tx_k) = p(Tx_{i-1},Tx_k)$$

$\leq \varphi\left(\frac{1}{2}\max\{p(x_{i-1},x_k),p(x_{i-1},Tx_{i-1}),p(x_k,Tx_k),p(x_{i-1},Tx_k),p(x_k,Tx_{i-1})\}\right)$.

Since $\{p(x_{i-1},x_k),p(x_{i-1},Tx_{i-1}),p(x_k,Tx_k),p(x_{i-1},Tx_k),p(x_k,Tx_{i-1})\}\subset A_n$, the definition of $diam(A_n)$ implies that

$\max\{p(x_{i-1},x_k),p(x_{i-1},Tx_{i-1}),p(x_k,Tx_k),p(x_{i-1},Tx_k),p(x_k,Tx_{i-1})\}\leq \alpha_n$.

Furthermore, from Definition 2.6.1 (φ is non-decreasing), we have $\alpha_n \leq \varphi(\alpha_n)$, which contradicts Definition 2.6.2. This contradiction proves that in this case $x_i = x_0$.

(b) If $Tx_{i-1}\notin K$, then by construction of $\{x_n\}$, $i\geq 2$ and $x_i \in seg[Tx_{i-2},Tx_{i-1}]\cap \partial K$. Since X is a partial metric space of hyperbolic type, Lemma 3.2. implies that for $x = Tx_{i-2}, y = Tx_{i-1}, z = x_i, u = Tx_k$ the following holds:

$p(Tx_k,x_i)\leq 2\max\{p(Tx_k,Tx_{i-2}),p(Tx_k,Tx_{i-1})\}$

If $\max\{p(Tx_k,Tx_{i-2}),p(Tx_k,Tx_{i-1})\} = p(Tx_k,Tx_{i-2})$, then condition (ii) and Definition 2.6.2 implies:

$\alpha_n = p(Tx_k,x_i)\leq 2p(Tx_k,Tx_{i-2})$

$\leq 2\varphi\left(\frac{1}{2}\max\{p(x_k,x_{i-2}),p(x_k,Tx_k),p(x_{i-2},Tx_{i-2}),p(x_{i-2},Tx_k),p(x_k,Tx_{i-1})\}\right)$

$< \max\{p(x_k,x_{i-2}),p(x_k,Tx_k),p(x_{i-2},Tx_{i-2}),p(x_{i-2},Tx_k),p(x_k,Tx_{i-1})\}$

$= \alpha_n$.

We have reached once again a contradiction.

Reasoning in the same way, if $\max\{p(Tx_k,T_{i-2}),p(Tx_k,Tx_{i-1}\ )\} = p(Tx_k,Tx_{i-1})$, we would reach to the same contradiction ($\alpha_n < \alpha_n$) and prove once again that $x_i = x_0$.

Thus we have shown that $\alpha_n = p(x_0,Tx_k)$.

Case 2. Let $\alpha_n = p(x_i,x_k)$ for some $0\leq i<k\leq n-1$. $k>0$ implies that $x_{k-1}\in K$ and Tx_{k-1} is defined.

(a) If $Tx_{k-1}\in K$, then $x_k = Tx_{k-1}$. Thus, Case 2(a) reduces to Case 1(a).
(b) If $Tx_{k-1}\notin K$, then $k\geq 2$ and $x_k\in seg[Tx_{k-2},Tx_{k-1}]\cap\partial K$. Similarly, Case 2(b) reduces to Case 1(b).

Case 3. Similarly $\alpha_n = p(Tx_i,Tx_k)$ is also impossible, because

$$\alpha_n = p(Tx_i,Tx_k)\leq \varphi\left(\frac{1}{2}\max\{p(x_i,x_k),p(x_i,Tx_i),p(x_k,Tx_k),p(x_i,Tx_k),p(x_k,Tx_i)\}\right)\leq \varphi\left(\frac{1}{2}\alpha_n\right)\leq \varphi(\alpha_n),$$

which contradicts Definition 2.6.2.

This way we have shown that $\alpha_n = \max\{p(x_0,Tx_k): k = 0,1,\dots,n-1\}$. To conclude the proof of this step we show that the sequence $\{A_n\}$ is bounded.

By definition $\{\alpha_n\}$ is non- decreasing. To prove that this sequence is bounded suffices to show that $\lim_{n\to\infty}\alpha_n < +\infty$.

Suppose to the contrary that $\lim_{n\to\infty}\alpha_n = +\infty$. From the conditions of Theorem 3.3 we have that $\lim_{n\to\infty}(t-\varphi(t)) = +\infty$, so there exists a positive number $\delta>0$ such that for all $t>\delta$, we have

$t-\varphi(t) > p(x_0,Tx_0) - p(Tx_0,Tx_0) > 0$.

By supposition, $\lim_{n\to\infty}\alpha_n = +\infty$. This means that for $\delta>0$ there exists an integer n such that $\alpha_n > \delta$, and

$\alpha_n - \varphi(\alpha_n) > p(x_0, Tx_0) - p(Tx_0, Tx_0)$.

For a fixed integer n such that $\alpha_n > \delta$, we know that $\alpha_n = p(x_0, Tx_{k(n)})$, for some $k(n) \in \{0,1, \dots, n-1\}$ and we consider the following cases.

Case 1. If $k(n) = 0$, then $\alpha_n = p(x_0, Tx_0)$, and so $\{x_n\}$ and $\{Tx_n\}$ are bounded.

Case 2. İf $k(n) > 0$, then $\alpha_n = p(x_0, Tx_{k(n)})$ and by using the triangle inequality, condition (ii), and the definition of φ it holds that:

$$\alpha_n = p(x_0, Tx_{k(n)})$$

$$\leq p(x_0, Tx_0) + p(Tx_0, Tx_{k(n)}) - p(Tx_0, Tx_0)$$

$$\leq p(x_0, Tx_0) + \varphi\left(\frac{1}{2}\max\{p(x_0, x_{k(n)}), p(x_0, Tx_0), p(x_{k(n)}, Tx_{k(n)}), p(x_0, Tx_{k(n)}), p(x_{k(n)}, Tx_0)\}\right) - p(Tx_0, Tx_0)$$

$$\alpha_n \leq p(x_0, Tx_0) + \varphi(\alpha_n) - p(Tx_0, Tx_0)$$

$$\alpha_n - \varphi(\alpha_n) \leq p(x_0, Tx_0) - p(Tx_0, Tx_0).$$

This contradiction implies that $\lim_{n\to\infty} \alpha_n = \alpha < +\infty$, and we have thus proved that $\{x_n\}$ and $\{Tx_n\}$ are bounded.

Step 3. İn this step we will show that both these sequences are Cauchy. We start by defining for $n \geq 2$,

$$B_n = \{x_i\}_{i\geq n} \cup \{Tx_i\}_{i\geq n} \text{ and } \beta_n = diam(B_n)$$

The sequence $\{B_n\}$ is Cauchy if $\lim_{n\to\infty} \beta_n = 0$. The definition of $\{\beta_n\}$ implies that this sequence is non-increasing and also it is bounded (because $\beta_n \geq 0$ for all $n \geq 2$). Thus $\{\beta_n\}$ is convergent and it converges to some point $\beta \geq 0$. To conclude the proof on this step we will show that $\beta = 0$. Let us suppose that $\beta > 0$.

On the other hand we have that $\beta_n = diam(B) = \sup\{p(x_i, Tx_j), p(x_i, x_j), p(Tx_i, Tx_j): i, j \geq n\}$.

Applying a method similar to that used in Step 2, we can show that

$$\beta_n = \sup\{p(x_n, Tx_k): k \geq n\}.$$

From the characteristic property of the supremum it holds that for every integer p there exist an index $k(p) > p$ such that $\beta_p - \frac{1}{p} < p(x_p, Tx_{k(p)}) < \beta_p$. Thus, by taking the limit, we have $\lim_{p\to\infty} p(x_p, Tx_{k(p)}) = \beta$.

Now we consider the following cases.

Case 1. If $x_p = Tx_{p-1}$, then by condition (ii) and Definition 2.6.1, it holds that:

$$p(x_p, Tx_{k(p)}) = p(Tx_{p-1}, Tx_{k(p)})$$
$$\leq \varphi\left(\frac{1}{2}\max\{p(x_{p-1}, x_{k(p)}), p(x_{p-1}, Tx_{p-1}), p(x_{k(p)}, Tx_{k(p)}), p(x_{p-1}, Tx_{k(p)}), p(x_{k(p)}, Tx_{p-1})\}\right)$$

$$\leq \varphi(\beta_n)$$

Since φ is a lower semi continuous function, by taking the limit for $p \to \infty$ we would have $\beta \leq \varphi(\beta)$, which is in contradiction with Definition 2.6.2. This means that in this case, $\beta = 0$.

Case 2. If $x_p \neq Tx_{p-1}$, then by construction of $\{x_n\}$ we have $x_p \in seg[Tx_{p-2}, Tx_{p-1}] \cap \partial K$. Since X is a partial metric space of hyperbolic type, Lemma 3.2 implies, for $x = T_{p-2}, y = Tx_{p-1}, z = x_p, u = Tx_{k(p)}$, that

$p\big(Tx_{k(p)}, x_p\big) \leq 2\max\{p\big(Tx_{k(p)}, Tx_{p-2}\big), p\big(Tx_{k(p)}, Tx_{p-1}\big)\}.$

If $\max\{p\big(Tx_{k(p)}, Tx_{p-2}\big), p\big(Tx_{k(p)}, Tx_{p-1}\big)\} = p\big(Tx_{k(p)}, Tx_{p-2}\big)$, then condition (ii) and Definition 2.6.1 implies:

$p\big(Tx_{k(p)}, x_p\big) \leq 2p\big(Tx_{k(p)}, Tx_{p-2}\big)$

$\leq 2\varphi\left(\frac{1}{2}\max\{p\big(x_{k(p)}, x_{p-2}\big), p\big(x_{k(p)}, Tx_{k(p)}\big), p\big(x_{p-2}, Tx_{p-2}\big), p\big(x_{p-2}, Tx_{k(p)}\big), p\big(x_{k(p)}, Tx_{p-2}\big)\}\right)$

$< \beta_n,$

which by taking the limit for $p \to \infty$, leads to the contradiction $\beta < \beta$.

If $\max\{p\big(Tx_{k(p)}, Tx_{p-2}\big), p\big(Tx_{k(p)}, Tx_{p-1}\big)\} = p\big(Tx_{k(p)}, Tx_{p-1}\big)$, by following the above reasoning we get once again the contradiction $\beta < \beta$. This contradiction proves again that $\beta = 0$, which in turn shows that the sequences $\{x_n\}$ and $\{Tx_n\}$ are both Cauchy.

Since X is complete and K is closed, both these sequences converge to some point $u \in K$, meaning:

$\lim_{n\to\infty} p(x_n, u) = p(u, u) = \lim_{n\to\infty} p(Tx_n, u).$

Step 4. In this step we show that u is a fixed point, i.e. $Tu = u \Leftrightarrow p(u, u) = p(u, Tu) = p(Tu, Tu)$.
Suppose the contrary, which based on Definition 2.1 means that $p(u, Tu) > 0$.

Let n be a fixed integer, and $x_n \in K$. Then the following cases are possible.

Case 1. If $Tx_n \in K$, then $p(Tx_n, Tu) \leq \varphi(\max\{p(x_n, u), p(x_n, Tx_n), p(u, Tu), p(u, Tx_n), p(x_n, Tu)\})$
By taking the limit we get the following contradiction

$p(u, Tu) \leq \varphi(\max\{p(u, u), p(u, Tu), p(u, Tu), p(u, u), p(u, Tu)\}) = \varphi\big(p(u, Tu)\big).$

Case 2. If $Tx_n \notin K$, then $x_{n+1} \in seg[Tx_{n-1}, Tx_n] \cap \partial K$ and because X is a partial metric space of hyperbolic type, by Lemma 3.2 it holds that

$p(Tu, x_{n+1}) \leq 2\max\{p(Tu, Tx_{n-1}), p(Tu, Tx_n)\}.$

If $\max\{p(Tu, x_{n-1}), p(Tu, Tx_n)\} = p(Tu, Tx_{n-1})$, then

$p(Tu, x_{n+1}) \leq 2\, p(Tu, Tx_{n-1}) \leq 2\varphi\left(\frac{1}{2}\max\{p(u, x_{n-1}), p(u, Tu), p(x_{n-1}, Tx_{n-1}), p(x_{n-1}, Tu), p(u, Tx_{n-1})\}\right)$

$< \max\{p(u, x_{n-1}), p(u, Tu), p(x_{n-1}, Tx_{n-1}), p(x_{n-1}, Tu), p(u, Tx_{n-1})\}.$

By taking the limit we get once more a contradiction as follows

$p(Tu, u) < \max\{p(u, u), p(u, Tu), p(u, Tu), p(u, Tu), p(u, Tu)\} = p(u, Tu).$

If $\max\{p(Tu, x_{n-1}), p(Tu, Tx_n)\} = p(Tu, Tx_n)$, then by following the same reasoning as above we get the same contradiction.

This contradiction shows that $p(u, u) = p(u, Tu) = p(Tu, Tu)$, and by definition of the partial metric we have $Tu = u$.

Step 5. In this step we prove the uniqueness of the fixed point u.

Suppose to the contrary that there exists $v \in K$, such that $u \neq v$ and $Tv = v$. Thus, by the definition of the partial metric we have that $p(u,u) < p(u,v)$, $p(v,v) < p(u,v)$ and $p(Tv,Tv) = p(Tv,v) = p(v,v)$. Then by applying condition (ii) and Definition 2.6.2 we get

$$p(u,v) = p(Tu,Tv)$$

$$\leq \varphi\left(\frac{1}{2}\max\{p(u,v), p(u,Tu), p(v,Tv), p(v,Tu), p(u,Tv)\}\right)$$

$$\leq \varphi(\max\{p(u,v), p(u,u), p(v,v), p(v,u), p(u,v)\})$$

$$= \varphi\big(p(u,v)\big).$$

This contradiction shows that $p(u,u) = p(u,v) = p(v,v)$, and by definition $u = v$. ■

Remark 1. Theorem 3.3 is a generalization of Theorem 2.1 from Ćirić (2006) for partial metric spaces of hyperbolic type. For the functions φ_i (where $i = 1,2,3,4,5$), in Theorem 2.1, we can find $\varphi = \max_{i=1,\dots,5}\{\varphi_i\}$, which also satisfies the same conditions as each of φ_i.

Remark 2. As mentioned in the preliminaries, several authors have proven interesting fixed point results using φ- contracting conditions. Theorem 3.3 shows that it is possible the generalize results gained from these type of conditions from metric spaces of hyperbolic type to partial metric spaces of hyperbolic type.

Remark 3. By taking $\varphi(t) = 2\lambda t$, where $0 < \lambda < 1$, we get the following corollary.

Corollary 3.4. Let (X,p) be a complete partial metric space of hyperbolic type, K a nonempty closed subset of X and $T: K \to X$ a non- self mapping such that $T(\partial K) \subseteq K$, and

$$p(Tx,Ty) \leq \lambda \cdot \max\{p(x,y), p(x,Tx), p(y,Ty), p(x,Ty), p(y,Tx)\}$$

Then T has a unique fixed point in K.

Example 3.5. Consider the complete metrically convex partial metric space (R^+, p) where $p(x,y) = \max\{x,y\}$, for all $x, y \geq 0$. This space is also of hyperbolic type because the derived metric space (R^+, d_p), where $d_p(x,y)$ is the usual metric, is a metric space of hyperbolic type. $K = [0,1]$ is a closed subset of R^+.

We define $T: [0,1] \to R^+$ such that for all $x \in [0,1]$,

$$T(x) = \frac{x^2}{2(1+x)}.$$

Next we define $\varphi: R^+ \to R^+$,

$$\varphi(t) = \begin{cases} \dfrac{3t}{4+2t}, & 0 \leq t \leq 1 \\ \dfrac{1}{2}t, & t > 1 \end{cases}.$$

We note that φ is non- decreasing, $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$. Which means φ is an ultra- alternating distance. Also φ is lower semi- continuous and $\lim_{n\to\infty}(t - \varphi(t)) = +\infty$ and thus it satisfies the conditions of Theorem 3.3.

Next we show that $T: [0,1] \to R^+$ also satisfies the conditions of Theorem 3.3. Since $T(0) = 0$ and $T(1) = \frac{1}{4}$, condition (i) is satisfied.

To prove the second condition we first evaluate $p(x,y), p(x,Tx), p(y,Ty), p(y,Tx), p(x,Ty), p(Tx,Ty)$.

Let $x, y \in [0,1]$ and for simplicity suppose that $x < y$, then $p(x, y) = y$. Moreover, by simple calculation we find

$$p(x, Tx) = \max\{x, Tx\} = \max\left\{x, \frac{x^2}{2(1+x)}\right\} = x, \qquad p(y, Ty) = \max\left\{y, \frac{y^2}{2(1+y)}\right\} = y,$$

$$p(y, Tx) = \max\{y, Tx\} = \max\left\{y, \frac{x^2}{2(1+x)}\right\} = y, \qquad p(x, Ty) = \max\left\{x, \frac{y^2}{2(1+y)}\right\} = x.$$

Hence, for $0 \le x < y \le 1$,

$$\varphi\left(\frac{1}{2}\max\{p(x,y), p(x,Tx), p(y,Ty), p(x,Ty), p(y,Tx)\}\right) = \varphi\left(\frac{1}{2}\max\{x,y\}\right) = \varphi(y) = \frac{3y}{2(4+2y)}.$$

Since T is monotone increasing on [0,1]. Then for $x, y \in [0; 1]$, such that $x < y$, we have $Tx < Ty$. And thus,

$$p(Tx, Ty) = \max\left\{\frac{x^2}{2(1+x)}, \frac{y^2}{2(1+y)}\right\} = \frac{y^2}{2(1+y)}$$

By combining all the above calculations, we get

$$p(Tx, Ty) = \frac{y^2}{2(1+y)} \le \frac{3y}{2(4+2y)} = \varphi\left(\frac{1}{2}\max\{p(x,y), p(x,Tx), p(y,Ty), p(x,Ty), p(y,Tx)\}\right),$$

on the account that for all $y \in [0,1]$

$$\frac{y^2}{2(1+y)} - \frac{3y}{2(4+2y)} = \frac{2y^3 + y^2 - 3y}{2(1+y)(4+2y)} = \frac{y(2y^2 + y - 3)}{2(1+y)(4+2y)} < 0.$$

We have hence shown that T also satisfies condition (ii).

Therefore T has a fixed point, which is $x = 0$.

Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

Acknowledgements or Notes

* This article was presented as an oral presentation at the International Conference on Basic Sciences, Engineering and Technology (www.icbaset.net) held in Alanya/Turkey on May 02-05, 2024.

References

Alghamdi, M. A., Shahzad, N., & Valero, Ó. (2013). Fixed point theorems in generalized metric spaces with applications to computer science. *Fixed Point Theory and Applications*, *2013*, 1-20.

Assad, N. A., & Kirk, W. A. (1972). Fixed point theorems for set-valued mappings of contractive type. *Pacific Journal of Mathematics*, *43*(3), 553–562.

Boyd, D. W., & Wong, J. S. W. (1969). On nonlinear contractions. *Proceedings of the American Mathematical Society*, *20*(2), 458–464.

Bugajewski, D., Maćkowiak, P., & Wang, R. (2022). On compactness and fixed point theorems in partial metric spaces. *Fixed Point Theory*, *23*(1), 163–178.

Bukatin M., Kopperman R., Matthews S., & Pajoohesh, H. (2009). Partial metric spaces. *The American Mathematical Monthly*, *116*(8), 708-718.

Carbone, A., Rhoades, B.E., & Singh, S.P. (1989). A fixed point theorem for generalized contraction map. *Indian Journal of Pure and Applied Mathematics, 20*(6), 543-548.

Ćirić, L. B. (2006). Contractive type non-self-mappings on metric spaces of hyperbolic type. *Journal of Mathematical Analysis and Applications, 317*(1), 28–42.

Ćirić, L. B. (1974). A generalization of Banach's contraction principle. *Proceedings of the American Mathematical Society, 45*(2), 267.

Han, S., Wu, J., & Zhang, D. (2017). Properties and principles on partial metric spaces. *Topology and Its Applications, 230,* 77–98.

Izadi, M. (2012). A fixed point theorem on quasi-metric spaces of hyperbolic type. *International Journal of Contemporary Mathematical Sciences, 7*(44), 2155- 2160.

Jachymski, J.R. (1994). Common fixed point theorems for some families of maps. *Indian Journal of Pure & Applied Mathematics, 25,* 925-934.

Kirk, W. A. (1982). Krasnoselskii's iteration process in hyperbolic space. *Numerical Functional Analysis and Optimization, 4*(4), 371–381.

Kumar, S., Rugumisa, T., & Imdad, M. (2017). Common fixed points in metrically convex partial metric spaces. *Konuralp Journal of Mathematics, 5*(2), 54-69.

Matthews, S. G. (1992). Partial metric spaces. *8th British Colloquium for Theoretical Computer Science. In Research Report, 212*, 708–718.

Matthews, S. (1994). Partial metric topology. *Annals of the New York Academy of Sciences, 728*(1), 183–197.

Park, S., & Rhoades, B.E. (1981). Meir-Keeler type contractive conditions. *Mathematica Japonica, 16*(1), 13-20.

Author Information

Silvja Cobani
"Fan S. Noli" University
Shëtitorja Rilindasit, Korçë 7001, Albania
Contact e-mail: *scobani@unkorce.edu.al*

Elida Hoxha
University of Tirana
Bulevardi Zogu I 25, Tirana 1016, Albania

To cite this article:

Cobani, S.,& Hoxha, E. (2024). A fixed point theorem on partial metric spaces of hyperbolic type. *The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM), 28,* 175-184.