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Application of the New General Integral Transform for Newton's Law of Cooling

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Abstract: Integral transformations are versatile mathematical techniques that are applicable in a wide range of applications in various fields in mathematics, engineering and applied sciences. Using integral transformations, complicated functions can be transformed into more simpler functions to be analyzed and to be solved. Integral transform methods have been extensively used to solve differential equation theories and numerical calculation. Newton's law of cooling is the fundamental law that describes the rate of heat transfer by a body to its surrounding through radiation. This important law can be modelled in the form of differential equation, which is solved by many researchers by using different methods. Some researchers used integral transforms to solve this equation. In this study, we use an integral transform, providing a more flexible and powerful approach for solving differential equations, called as "New General Integral Transform" as a generalization of the Laplace transform method and some applications are given to demonstrate the effectiveness of this transform.

Keywords: Newton's law of cooling, Differential equation, The new general integral transform

Introduction

Although he is best known for his laws of motion and universal gravitation, Newton's discovery of heat transfer laid the foundation for modern thermodynamics. His recognition of the relationship between temperature differences and heat flow ushered in a new era in scientific research, paving the way for a deeper understanding of energy dynamics.

Newton's law of cooling is a physical law that describes the rate of heat loss of an object to its surroundings. It states that the rate of heat loss is directly proportional to the temperature difference between the object and its surroundings. Newton's law of cooling is a special case of Stefan-Boltzmann's Law for small temperature differences (Jiji, 2009; Winterton, 1999; Baehr & Stephan, 2011). The accurate solution of Newton's law of cooling is crucial in various scientific and engineering applications, such as in the design of cooling systems, thermal management in electronics, and climate modeling. This important law can be modelled in the form of differential equation, which is solved by many researchers by using different methods. Some researchers used integral transforms to solve this equation. In this study, we use an integral transform, providing a more flexible and powerful approach to solving differential equations, called as "New General Integral Transform" which was defined by Jafari as a generalization of the Laplace transform method (Jafari, 2021).

Integral transforms are powerful mathematical tools used in the analysis and solution of differential equations. By transforming a differential equation into a different domain, such as the frequency domain or the Laplace domain, integral transforms simplify the problem-solving process and provide new insights into the dynamics of the system (Debnath & Bhatta, 2007).

Newton's Law of Cooling

Newton's law of cooling, which has a very important position in physics, is also modeled by differential equations. This law explains how the temperature change in an object occurs depending on the difference between the temperature of the object and the temperature of the environment in which the object is located. Newton's law of cooling states that a hot body releases heat energy into its environment (Jiji, 2009; Winterton, 1999; Baehr & Stephan, 2011). The amount of this energy emitted depends on the temperature difference between the object placed in the environment and the environment. While this energy transfer is taking place, as the temperature difference decreases, the energy transfer decreases and eventually the hot body equalizes with the ambient temperature (Jiji, 2009; Winterton, 1999; Baehr & Stephan, 2011). Newton's law of cooling is generally modeled by the linear ordinary differential equation (Jiji, 2009)

$$\frac{dT(t)}{dt} = -C(T(t) - T_e) \tag{1}$$

with the initial condition

$$T(t_0) = T_0$$

where T is the temperature of the substance, T_e is the constant temperature of the environment, T_0 is the initial temperature of the substance at time t_0 and C is the proportionality constant, known as convective heat transfer coefficient.

This seemingly modest equation reveals the complex interplay between thermal gradients and heat exchange, providing invaluable insight into countless real-world scenarios from cooling a freshly brewed cup of coffee to regulating spacecraft temperatures in outer space. Newton's law of cooling provides a versatile framework for analyzing and predicting thermal behavior.

Method

A New General Integral Transform

In this part, a new general integral transform which comprises most of the integral transform in the family of Laplace transform.

Definition 1. (Jafari, 2021) Let $f(t)$ be an integrable function defined for $t \geq 0$, $p(s)$ and $q(s)$ be positive real functions. Then, the new general integral transform $\mathcal{T}(s)$ of $f(t)$ is defined as follows,

$$T\{f(t);s\} = \mathcal{T}(s) = p(s)\int_0^\infty f(t)e^{-q(s)t}dt,$$

provided the integral exists for some $q(s)$. It is essential to note that this transform for those $f(t)$ is not continuously differentiable contains terms with negative or fractional powers of $q(s)$.

Theorem 1 (Existence Theorem). (Jafari, 2021) If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geq 0$ and is of exponential order *k*, that is $|f(t)| \leq M\,e^{kt}$ for some positive real number *M* and *k*, then $\mathcal{T}(s)$ exists for all $q(s) > k$.

Proof: Since $f(t)$ is piecewise continuous, $e^{-st}f(t)$ is integrable over any finite interval on the *t*-axis. Assume that $q(s) > k$. Then, the proof of the existence of the $\mathcal{T}(s)$ is obtained as follows:

$$\left\| \mathcal{T}(s) \right\| = \left\| T\{f(t);s\} \right\| = \left| p(s) \int_0^\infty f(t) e^{-q(s)t} dt \right| \le p(s) \int_0^\infty M e^{kt} e^{-q(s)t} dt = \frac{M\, p(s)}{q(s)-k}.$$

In Table1, one can find the transformation of some fundamental functions.

Table 1. The new general integral transform of some fundamental functions

f(t)	$\mathcal{T}(s)$
1	$\frac{p(s)}{q(s)}$
t	$\frac{p(s)}{(q(s))^2}$
t^α , $\alpha > 0$	$\frac{\Gamma(\alpha+1)\, p(s)}{(q(s))^{\alpha+1}}$, $\alpha > 0$
$e^{\alpha t}$	$\frac{p(s)}{q(s)-\alpha}$, $q(s) > \alpha$
$\sin(ct)$	$\frac{c\, p(s)}{c^2 + (q(s))^2}$
$\cos(ct)$	$\frac{p(s)\, q(s)}{c^2 + (q(s))^2}$

Theorem 2 (Transform of Derivatives). (Jafari, 2021) Let $f(t)$ be differentiable and $p(s)$ and $q(s)$ be positive real functions. Then,

1. $T\{f'(t);s\} = q(s)\, T\{f(t);s\} - p(s) f(0)$
2. $T\{f''(t);s\} = q^2(s)\, T\{f(t);s\} - q(s)p(s)f(0) - p(s)f'(0)$
3. $T\{f^{(n)}(t);s\} = q^n(s)\, T\{f(t);s\} - p(s)\sum_{k=0}^{n-1} q^{n-1-k}(s) f^{(k)}(0)$

Application to Newton's Law of Cooling

In this section, the new general integral transform will be applied to Newton's law of cooling equation. Firstly, taking the transform of the equation (1), we obtain

$$\mathcal{T}\left[\frac{dT(t)}{dt}\right] = \mathcal{T}\left[-C(T(t) - T_e)\right] \tag{2}$$

Now, rearranging the equation (2) according to the theorem 2 and Table 1, and substituting the value in the initial condition into this equation, we obtain

$$q(s)T(s) - p(s)T_0 = -C\left[T(s) - T_e \frac{p(s)}{q(s)}\right] \tag{3}$$

where $\mathcal{T}(s) = \mathcal{T}\left[T(t)\right]$.

If we make the equation (3) suitable for applying the inverse transform, we get

$$\mathcal{T}(s) = T_0 \frac{p(s)}{q(s)+c} + cT_e \frac{p(s)}{q(s)[q(s)+c]} \tag{4}$$

Finally, applying the inverse transform to the equation (4) using Table 1, we find the solution of Newton's law of cooling equation as

$$T(t) = T_e + (T_0 - T_e)e^{-ct}\ .$$

We will examine the effectiveness of this method through some numerical applications and compare the results with those found with other methods in the literature.

Application 1. (Patil et al., 2022c) A hot milk with initial temperature $115°\ C$ is kept in an environment with temperature $35°\ C$. The rate of temperature change is $20°\ C$ per/min, how long will it take for this milk to cool down to temperature $40°\ C$?

Assuming that milk obeys Newton's law of cooling, we arrange the equation (1) according to the values given in the question as

$$\frac{dT(t)}{dt} = -C(T-35)$$

with the initial conditions

$$T(0) = 115, \quad T'(0) = -20$$

First, using the initial conditions in the above equation, we get the value of C as

$$-20 = -C(115-35)$$
$$C = 0.25$$

Substituting this C value into the above equation, we get

$$\frac{dT(t)}{dt} = -0.25(T-35)$$

Now, applying the transform to both sides of the above equation, we obtain

$$\mathcal{T}\left[\frac{dT(t)}{dt}\right] = -0.25\mathcal{T}\left[T-35\right]$$

Rearranging the last equation according to the theorem 2 and Table 1, we get

$$q(s)T(s) - 115p(s) = -0.25\left[T(s) - 35\frac{p(s)}{q(s)}\right]$$

Rearranging this equation, we get

$$\mathcal{T}(s) = 115\frac{p(s)}{q(s)+0.25} + (0.25)(35)\frac{p(s)}{q(s)[q(s)+0.25]}$$

and finally, applying the inverse transform to this equation, we obtain the solution as

$$T(t) = 80e^{-0.25t} + 35$$

We can find out how long it will take for the milk to cool down to $40°\ C$ based on our solution as

$$
\begin{aligned}
40 &= 35 + 80e^{-0.25t} \\
80e^{-0.25t} &= 5 \\
e^{0.25t} &= 16 \\
0.25t &= \ln 16 \\
t &= 11.090354889
\end{aligned}
$$

This result is in good agreement with the results obtained by other methods (Peker et al., 2024; Patil et al.,2022a, 2022b, 2022c).

Application 2. (Patil et al., 2022c) The heated iron with an initial temperature of $50^{\circ}\ C$ is kept in an environment with temperature of $27^{\circ}\ C$. Since the rate of temperature change is $3^{\circ}\ C$ per/min, how long will it take for this iron to cool down to temperature $36^{\circ}\ C$?

Assuming that iron obeys Newton's law of cooling, we arrange the equation (1) according to the values given in the question as

$$\frac{dT(t)}{dt} = -C(T - 27)$$

with the initial conditions

$$T(0) = 50, \quad T'(0) = -3$$

First, using the initial conditions expressed above, we find the value of C as

$$
\begin{aligned}
-3 &= -C(50 - 27) \\
C &= 0.13
\end{aligned}
$$

Substituting this C value, we get

$$\frac{dT(t)}{dt} = -0.13(T - 27)$$

Now, applying the transform to both sides of the above equation, we obtain

$$\mathcal{T}\left[\frac{dT(t)}{dt}\right] = -0.13\mathcal{T}\left[T - 27\right]$$

Rearranging the last equation according to the theorem 2 and Table 1, we get

$$q(s)T(s) - 50p(s) = -0.13\left[T(s) - 27\frac{p(s)}{q(s)}\right]$$

Rearranging this equation, we get

$$\mathcal{T}(s) = 50\frac{p(s)}{q(s) + 0.13} + (0.13)(27)\frac{p(s)}{q(s)[q(s) + 0.13]}$$

Finally, applying the inverse transform to this equation, we find the solution as

$$T(t) = 23e^{-0.13t} + 27$$

Now we can find out how long it will take for the milk to cool down to $36^{\circ}\ C$ using our solution as follows:

$$36 = 27 + 23e^{-0.13t}$$
$$23e^{-0.13t} = 9$$
$$e^{0.13t} = \frac{23}{9}$$
$$0.13t = \ln\left(\frac{23}{9}\right)$$
$$t = 7.2175$$

This result coincides with the ones found by other methods (Peker et al., 2024; Patil et al.,2022a, 2022b, 2022c).

Application 3. (Naresh, 2017) While the ambient temperature is $20^\circ\ C$, the temperature of the water drops from $100^\circ\ C$ to $80^\circ\ C$ in 20 minutes. What will be the temperature after 30 minutes and how long will it take for this water to cool to $45^\circ\ C$?

Assuming that water obeys Newton's law of cooling, we arrange the equation (1) according to the values given in the question as

$$\frac{dT(t)}{dt} = -C(T-20)$$

with the initial conditions

$$T(0) = 100, \quad T(20) = 80$$

First, applying the transform to both sides of the above equation, we obtain

$$\mathcal{T}\left[\frac{dT(t)}{dt}\right] = -C\mathcal{T}\left[T-20\right]$$

Rearranging the last equation according to the theorem 2 and Table 1, we get

$$q(s)T(s) - 100p(s) = -C\left[T(s) - 20\frac{p(s)}{q(s)}\right]$$

Rearranging this equation, we get

$$\mathcal{T}(s) = 100\frac{p(s)}{q(s)+C} + (C)(20)\frac{p(s)}{q(s)[q(s)+C]}$$

Finally, applying the inverse transform to this equation, we find the solution equation as

$$T(t) = 80e^{-Ct} + 20$$

Now let's find the value of C. If we use $T(20) = 80^\circ$ in the last equation, we obtain

$$80 = 20 + 80e^{-20C}$$
$$e^{-c} = \left(\frac{3}{4}\right)^{\frac{1}{20}}$$

Using this result to find $T(30)$, we obtain

$$T(30) = 20 + 80e^{-30c}$$

$$T(30) = 20 + 80\left(\frac{3}{4}\right)^{\frac{3}{2}} = 71.96°C$$

Now we can find out how long it will take for the water to cool down to $45°\ C$ using the solution as follows:

$$45 = 20 + 80(e^{-C})^{t}$$

$$\left(\frac{3}{4}\right)^{\frac{t}{20}} = \frac{25}{80}$$

$$t = 80.8636283723$$

This result is in good agreement with the results obtained by other methods (Naresh, 2017; Peker et al., 2024).

Conclusion

In conclusion, the accurate solution of Newton's law of cooling is essential for understanding heat transfer phenomena and optimizing engineering applications. By introducing the new integral transform method for solving differential equations, researchers can enhance the efficiency and accuracy of solving Newton's law of cooling. By leveraging the unique properties of integral transforms and the innovative approach of the new integral transform method, scientists and engineers can gain deeper insights into the dynamics of heat transfer processes and improve the performance of cooling systems. The application of the new integral transform method represents a significant advancement in the field of mathematical analysis and holds promise for revolutionizing the solution of complex differential equations in various scientific and engineering disciplines.

Recommendation

This new integral transform can be applied to different models appearing in other applied sciences so that revealing the efficiency of this transform.

Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

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