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# On Mersenne Power GCD Matrices

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**Abstract**: This paper explores the $n \times n$ Mersenne power GCD matrices defined on sets of positive integers, focusing on factor-closed and gcd-closed sets. By employing the form $f(t_i, t_j) = 2^{(t_i,t_j)} - 1$, we investigate the $r^{th}$ power Mersenne GCD matrix $(M^r)$ and provide comprehensive insights into its factorizations, determinants, reciprocals, and inverses. Building upon previous research, particularly Chun's work on power GCD matrices, we extend the analysis to Mersenne numbers, offering a thorough understanding of their properties. The study contributes to the broader understanding of arithmetic functions and their applications in matrix theory.

**Keywords:** Power GCD Matrix, Factor-closed sets, GCD-closed sets, Mersenne numbers.

## Introduction and Preliminaries

Let $T = \{t_1, t_2, \ldots, t_n\}$ be a well-ordered set of $n$ distinct positive integers with $t_1 < t_2 < \ldots < t_n$. The power GCD matrix on $T$ is also $n \times n$ square matrix such that $(T^r)_{ij} = (t_i, t_j)^r$, where where $(t_i, t_j)$ is the greatest common divisor of $t_i$ and $t_j$ and $r$ is any real number. Set $T$ is said to be factor-closed if $t_k \in T$ for any divisor $t_k$ of $t_i \in T$, and is gcd-closed if $(t_i, t_j) \in T$, for every $t_i$ and $t_j$in $T$.

In his work published in Smith(1875,1876) demonstrated that for a factor-closed set $T = \{t_1, t_2, \ldots, t_n\}$ of distinct positive integers, the determinant $det(T) = \varphi(t_1)\varphi(t_2)\ldots\varphi(t_n)$ (Smith, 1876). Subsequent to Smith's findings, numerous studies have contributed to the understanding of GCD and LCM matrices, including works by Beslin and Ligh (1989, 1991, 1992), ElKassar et al. (2009, 2010) and Awad et al. (2019, 2020, 2023), and others. In 1996, Chun introduced the concept of $r^{th}$ power GCD matrices on both factor-closed and gcd-closed sets for any real number $r$ (Chun, 1996). He determined their determinants, inverses, and reciprocals over the domain of natural numbers. Let $f$ be an arithmetical function of the form $f(n) = \sum_{d|n} g(d)$. The matrix $[f(i,j)]_{n\times n}$ given by the value of $f$ in greatest common divisor of $(i,j)$, $f(i,j)$ as its $i,j$ entry is called the greatest common divisor (GCD) matrix. In 2010, (Bege, 2010) considered the generalization of this matrix where the elements are in the form $f(i,(i,j))$ considered a generalization of the GCD matrices, where the elements are of the form $[f(i,j)]_{n\times n}$.

Inspired by the above works, we use the special form $f(t_i, t_j) = 2^{(t_i,t_j)} - 1$ in order to study the $r^{th}$ power Mersenne GCD matrix $(M^r)$ on the both cases for $T$ as factor-closed and as gcd-closed set. In addition, we give a full description of its factorizations, determinants, reciprocals, and inverses.



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## Preliminaries

In the following, we consider $T = \{t_1, t_2, \ldots, t_n\}$ as an arbitrary set of distinct positive natural integers, and $r$ is any real number.

**Definition 1.** The Mersenne $r^{th}$ power GCD matrix $(M^r)$ defined on $T$ is the $n \times n$ square matrix whose $ij^{th}$ entries are of the form

$$(m_{ij})^r = \left(2^{(t_i,t_j)} - 1\right)^r.$$

**Definition 2.** The generalized Mersenne power function $g(t)$ on $T$ is defined inductively for all $1 \le i \le n$ as

$$g(t_i) = \sum_{d \mid t_i} (2^n - 1)^r \, \mu(t_i/d).$$

**Definition 3.** The reciprocal of Mersenne power $GCD$ matrix is the $n \times n$ matrix $M^{-r}$ such that

$$(M^{-r})_{ij} = \frac{1}{(f_{ij})^r} = \frac{1}{\left(2^{(t_i,t_j)} - 1\right)^r}.$$

**Definition 4.** The generalized reciprocal Mersenne power function on $T$ is defined inductively for all $1 \le i \le n$ as

$$h(t) = \sum_{d|t_i} \left(\frac{1}{(2^d - 1)}\right)^r \mu(t_i\,/d).$$

**Definition 5.** The inverse of Mersenne power $GCD$ matrix is the square $n \times n$ matrix $(M^r)^{-1}$ such that $(M^r)(M^r)^{-1} = I_n$.

## Main Results

## Mersenne Power GCD Matrices Defined on Arbitrary Sets

In the following, we study Mersenne power $GCD$ matrices defined on arbitrary sets that are either factor-closed or not. A complete characterization is also given.

## Structure Theorems

**Theorem 1.** Let $T = \{t_1, t_2, \ldots, t_n\}$ be a non factor-closed set of positive integers and $\overline{T} = \{y_1, y_2, \ldots, y_m\}$ be the factor-closed closure of $T$ (the minimal factor-closed set containing $T$). Then, $(M^r) = EA_rE^T$, where $E$ is the $n \times m$ incidence matrix of $\overline{T}$ on $T$, and $A_r$ is the $m \times m$ diagonal matrix such that $a_{ii} = g(y_i)$.

*Proof.* Since $E$ is an $n \times m$ incidence matrix of $\overline{T}$ on $T$, then $e_{ij} = 1$ if $y_j \mid t_i$ and 0 otherwise. Hence,

$$\begin{aligned}(EA_rE^T)_{ij} &= \sum_{k=1}^{n} \left(e_{ik}a_{kk}e_{kj}\right) = \sum_{y_k \mid t_i,\, y_k \mid t_j} g\,(y_k) = \sum_{y_k \mid (t_i,t_j)} g\,(y_k) \\ &= \sum_{y_k \mid (t_i,t_j)} \sum_{d \mid y_k} (2^d - 1)^r \, \mu\left(\frac{y_k}{d}\right) = \sum_{d \mid y_k} \mu\left(\frac{y_k}{d}\right) \sum_{y_k \mid (t_i,t_j)} (2^d - 1)^r \\ &= \sum_{d \mid (t_i,t_j)} (2^d - 1)^r = \left(2^{(t_i,t_j)} - 1\right)^r.\end{aligned}$$

**Theorem 2.** Let $T = \{t_1, t_2, \ldots, t_n\}$ be a non factor-closed set of positive integers, and $\overline{T} = \{y_1, y_2, \ldots, y_m\}$ be the factor-closed closure of $T$. Then, $(M^r) = A_r E^T$ where $a_{ij} = g(y_j)$ if $y_j \mid t_i$ and 0 otherwise, and $E$ is the corresponding incidence matrix relative to $A_r$.

*Proof.* The $ij^{th}$ entries of the incidence matrix $E$ relative to $A_r$ are defined as: $e_{ij} = 1$ if $a_{ij} \neq 0$ and 0 otherwise. So, the $ij^{th}$ entry of $A_r E^T$ is

$$(A_r E^T)_{ij} = \sum_{k=1}^{n} \left(a_{ik} e_{kj}\right) = \sum_{y_k \mid t_i,\, y_k \mid t_j} g\,(y_k) = \sum_{y_k \mid (t_i, t_j)} g\,(y_k) = \left(2^{(t_i, t_j)} - 1\right)^r.$$

**Theorem 3.** Let $T = \{t_1, t_2, \ldots, t_n\}$ be a non factor-closed set of positive integers, and $\overline{T} = \{y_1, y_2, \ldots, y_m\}$ be the factor-closed closure of $T$. Then, $(M^r) = A_r A_r^T$ where $a_{ij} = \sqrt{g(y_j)}$ if $y_j \mid t_i$ and 0 otherwise.
*Proof.* The $ij^{th}$ entry of $A_r A_r^T$ is defined as:

$$(A_r A_r^T)_{ij} = \sum_{k=1}^{n} \left(a_{ik} a_{kj}\right) = \sum_{\substack{y_k \mid t_i \\ y_k \mid t_j}} \sqrt{g(y_k)}\sqrt{g(y_k)} = \sum_{y_k \mid (t_i, t_j)} g\,(y_k) = \left(2^{(t_i, t_j)} - 1\right)^r.$$

## Determinants

**Theorem 4.** Let $T = \{t_1, t_2, \ldots, t_n\}$ be a non factor-closed set of positive integers, and $\overline{T} = \{y_1, y_2, \ldots, y_m\}$ be the factor-closed closure of $T$ with $n \leq m$. Let $E_{(k_1, k_2, \ldots, k_n)_r}$ be the submatrix consisting of the $k_1^{th}, k_2^{th}, \ldots, k_n^{th}$ columns of $E$ for some indices $k_i$ such that $1 \leq k_1 < \ldots < k_n \leq m$. Then,

$$\det(M^r) = \sum_{1 \leq k_1 < \ldots < k_n \leq m} \left( \left(\det E_{(k_1, k_2, \ldots, k_n)_r}\right)^2 \prod_{i=1}^{n} \left( \sum_{d \mid t_i} \left(2^d - 1\right)^r \mu\left(\frac{t_i}{d}\right) \right) \right).$$

*Proof.* Let $A_r = \left(a_{ij}\right)_{m \times m}$ be defined as $a_{ij} = \sum_{d \mid t_i} \left(2^d - 1\right)^r \mu(t_i/d)$ if $y_j \mid t_i$ and 0 otherwise, and let $E = \left(e_{ij}\right)$ be the corresponding incidence matrix relative to $A_r$. Since $A_r$ is a triangular matrix with $a_{ii} = \sum_{d \mid t_i} \left(2^d - 1\right)^r \mu(t_i/d)$ for all $1 \leq i \leq m$, then the $ij^{th}$ entry of $A_r$ can be written as $a_{ij} = e_{ij} \sum_{d \mid t_i} \left(2^d - 1\right)^r \mu(t_i/d)$. Define, for some indices $k_i$ such that $1 \leq k_1 < \ldots < k_n \leq m$, the matrices $A_{(k_1, k_2, \ldots, k_n)}$ and $E_{(k_1, k_2, \ldots, k_n)}$ to be the submatrices consisting of the $k_1^{th}, k_2^{th}, \ldots, k_n^{th}$ columns of $A$ and $E$, respectively. Then, $A_{(k_1, k_2, \ldots, k_n)} = E_{(k_1, k_2, \ldots, k_n)} D_{A_r}$, where $D_{A_r}$ is the $n \times n$ diagonal submatrix of $A_r$ whose diagonal entries are $d_{ii} = \sum_{d \mid t_i} \left(2^d - 1\right)^r \mu(t_i/d)$. Therefore,

$$\det\left(A_{r_{(k_1, k_2, \ldots, k_n)}}\right) = \det\left(E_{(k_1, k_2, \ldots, k_n)}\right) \left(\prod_{i=1}^{n} d_{ii}\right).$$

Applying Cauchy-Binet formula, we obtain

$$\begin{aligned}
\det(M^r) &= \det(A_r E^T) \\
&= \sum_{1 \leq k_1 < k_2 < \ldots < k_n \leq m} \left(\det A_{r_{(k_1, k_2, \ldots, k_n)}}\right) \left(\det E_{(k_1, k_2, \ldots, k_n)}\right)^T \\
&= \sum_{1 \leq k_1 < \ldots < k_n \leq m} \det\left(E_{r_{(k_1, \ldots, k_n)}}\right) \sum_{d \mid t_i} \left(2^d - 1\right)^r \mu\left(\frac{t_i}{d}\right) \left(\det E_{r_{(k_1, \ldots, k_n)}}\right)^T \\
&= \sum_{1 \leq k_1 < k_2 < \ldots < k_n \leq m} \left( \sum_{d \mid t_i} \left(2^d - 1\right)^r \mu\left(\frac{t_i}{d}\right) \right) \left(\det E_{r_{(k_1, k_2, \ldots, k_n)}}\right)^2.
\end{aligned}$$

**Example 1.:** Consider the non factor-closed set $T = \{2,3,4\}$ and its factor closure $\overline{T} = \{1,2,3,4\}$. Then, the $2^{nd}$ power Mersenne $GCD$ matrix defined on $T$ is:

$$M^2 = \begin{bmatrix} 17^2 & 5^2 & 17^2 \\ 5^2 & 257^2 & 5^2 \\ 17^2 & 5^2 & 65537^2 \end{bmatrix} = \begin{bmatrix} 289 & 25 & 289 \\ 25 & 66049 & 25 \\ 289 & 25 & 4295098369 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} g(1) & 0 & 0 & 0 \\ 0 & g(2) & 0 & 0 \\ 0 & 0 & g(3) & 0 \\ 0 & 0 & 0 & g(4) \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 & 0 \\ 0 & 264 & 0 & 0 \\ 0 & 0 & 66024 & 0 \\ 0 & 0 & 0 & 4295098080 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, E_{123} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, E_{124} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, E_{134} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \text{and } E_{234} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

It is clear that $M^2 = EA_2E^T$. Applying Cauchy-Binet formula, we get

$$\begin{aligned} \det(M^2) &= \sum_{1\le k_1<k_2<k_3\le 4} \left( \det E_{(k_1,k_2,k_3)} \prod_{i=1}^{4} \big(g(t_i)\big) \right) \\ &= g(1)g(2)g(3)[\det(E_{123})]^2 + g(1)g(2)g(4)[\det(E_{124})]^2 \\ &+ g(1)g(3)g(4)[\det(E_{134})]^2 + g(2)g(3)g(4)[\det(E_{234})]^2 \\ &= 0 + 28347647328000 + 7089488890848000 + 74865002687354880 \\ &= 81982839225530880 \end{aligned}$$

In the case where $T = \{t_1, t_2, \dots, t_n\}$ is a factor-closed set of distinct positive integers, we have the following corollary.

**Corollary 1.** If $T = \{t_1, t_2, \dots, t_n\}$ is a factor-closed set of distinct positive integers, then $A$ is $n \times n$ diagonal matrix with diagonal entries $a_{ii} = g(t_i)$ and $E$ is also $n \times n$ square incidence matrix relative to $A$, and hence

$$\det(M^r) = \prod_{i=1}^{n} g(t_i)$$

*Proof.* By Theorem 1, we have

$$\det[(M^r)] = \det(EA_rE^T) = \det(E)\det(A_r)\det(E^T) = 1 \times \det(A_r) \times 1 = \prod_{i=1}^{n} g(t_i).$$

By Theorem 2, we have

$$\det[(M^r)] = \det(A_rE^T) = \det(A_r)\det(E^T) = \det(A_r) = \prod_{i=1}^{n} g(t_i).$$

By Theorem 3, we have

$$\det[(M^r)] = \det(A_rA_r^T) = \det(A_r)\det(A_r^T) = \left(\prod_{i=1}^{n} \sqrt{g(t_i)}\right)\left(\prod_{i=1}^{n} \sqrt{g(t_k)}\right) = \prod_{i=1}^{n} g(t_i).$$

## Reciprocals and Inverses

**Theorem 5.** Let $T = \{t_1, t_2, \dots, t_n\}$ be a non factor-closed set of distinct positive integers. Then, $[(M^{-r})] = EA_{-r}E^T$, where $A_{-r} = diag\big(h(t_1), h(t_2), \dots, h(t_n)\big)$ and $E$ is an incidence matrix of $T$, such that $e_{ij} = 1$ if $t_j | t_i$ and 0 otherwise.

*Proof.* Let $\overline{T} = \{y_1, y_2, \ldots, y_m\}$ be the factor closed closure of $T$. Define the $m \times m$ diagonal matrix whose diagonal entries are $a_{ii} = h(y_i)$ for all $1 \le i \le m$. Let $E$ be the $n \times m$ incidence matrix of $\overline{T}$ relative to $T$ such that $e_{ij} = 1$ if $y_j | t_i$ and 0 otherwise. Then,

$$\begin{aligned}(EA_{-r}E^T)_{ij} &= \sum_{k=1}^{n} \left(e_{ik}a_{kk}e_{jk}\right) = \sum_{\substack{y_k|t_i \\ y_k|t_j}} h(y_k) = \sum_{y_k|(t_i,t_j)} h\,(y_k) \\ &= \sum_{y_k|(t_i,t_j)} \sum_{d\,|\,y_k} \frac{1}{\left(2^d-1\right)^r}\mu\left(\frac{y_k}{d}\right) = \frac{1}{\left(2^{(t_i,t_j)}-1\right)^r}.\end{aligned}$$

**Theorem 6.** Let $T = \{t_1, t_2, \ldots, t_n\}$ be a factor-closed set of distinct positive integers, and let $E$ be the incidence matrix relative to $T$, such that $e_{ij} = 1$ if $t_j | t_i$ and 0 otherwise. Then, the inverse of $E$ is the matrix $F^T$ such that $f_{ij} = \mu\left(\frac{t_i}{t_j}\right)$ if $t_j | t_i$ and 0 otherwise. Moreover,

$$(M^r)^{-1} = FA_r^{-1}F^T$$

*Proof.* Since $T$ is factor-closed, then $E$ is an $n \times n$ square invertible matrix such that

$$(EF^T) = \sum_{k=1}^{n} \left(e_{ik}f_{kj}\right) = \sum_{t_k|t_i} \left(f_{kj}\right) = \begin{cases} \sum_{t/\frac{t_i}{t_j}} (\mu(t)) & if\ t_j|t_i \\ 0 & otherwise \end{cases} = \begin{cases} 1 & if\ t_j|t_i \\ 0 & otherwise \end{cases}$$

This implies that $E^{-1} = F^T$, and hence

$$(M_r)^{-1} = (EA_rE^T)^{-1} = (E^{-1})^T(A_r)^{-1}(E)^{-1} = FA_r^{-1}F^T.$$

## Mersenne Power GCD Matrices Defined on Non gcd-closed Sets

In this section, we study Mersenne Power $GCD$ matrices defined on non gcd-closed sets. Full description of their factorizations, determinants, reciprocals, and inverses are given.

## Structure Theorems

We prove three different factorizations for Mersenne power $GCD$ matrices over non gcd-closed sets.

**Theorem 7.** Let $T = \{t_1, t_2, \ldots, t_n\}$ be a gcd-closed set of distinct positive integers, and let $g(t_k) = \sum_{d\,|\,t_k} \left(2^d - 1\right)^r \mu\left(\frac{t_k}{d}\right)$. Then,

$$\sum_{t_k|(t_i,t_j)} \left( \sum_{t_k|t_j, t_k\nmid t_u,\, t_u<t_j} g\,(t_k) \right) = \left(m_{ij}\right)^r$$

*Proof.* It is clear that any set $T$ of distinct positive integers is contained in a gcd-closed set. Denote by $\overline{T}$ to be the minimal gcd-closed set containing $T$. It is worthwise to observe that $\overline{T}$ usually contains considerably fewer elements than any factor-closed set containing $T$. Also, it is clear that $\sum_{t_k|t_j, t_k\nmid t_u,\, t_u<t_j} g\,(t_k)$ is not representative and counted only once and it is equal to $g(t_k)$. Therefore,

$$\sum_{t_k|(t_i,t_j)} \left( \sum_{t_k|t_j,\, t_k\nmid t_u,\, t_u<t_j} g\,(t_k) \right) = \sum_{t_k|(t_i,t_j)} g\,(t_k) = \left(m_{ij}\right)^r.$$

**Thereom 8.** Let $T = \{t_1, t_2, \dots, t_n\}$ be a non gcd-closed set of distinct positive integers, and $\overline{T} = \{y_1, y_2, \dots, y_m\}$ be the minimal gcd-closed set containing $T$, then $(F_r) = E A_r E^T$, where $E$ is the incidence matrix relative to $T$ and $A_r$ is an $m \times m$ diagonal matrix.

*Proof.* Let $\overline{T} = \{y_1, y_2, \dots, y_m\}$ be the minimal gcd-closed set containing $T$. Define the $m \times m$ diagonal matrix $A_r$ as follows:

$$A_r = diag\left(\sum_{\substack{\mathrm{d}|y_1 \\ \mathrm{d}|y_u \\ yu|y_1}} g(\mathrm{d}), \sum_{\substack{\mathrm{d}|y_2 \\ \mathrm{d}|y_u \\ yu|y_2}} g(\mathrm{d}), \dots, \sum_{\substack{\mathrm{d}|y_m \\ \mathrm{d}|y_u \\ yu|y_m}} g(\mathrm{d})\right)$$

where $g(n) = \sum_{d|n} \left(2^d - 1\right)^r \mu\left(\frac{n}{d}\right)$. Let $E$ be the incidence matrix of $\overline{T}$ on $T$ such that $e_{ij} = 1$ if $y_j | t_i$ and 0 otherwise. Then,

$$(E A_r E^T)_{ij} = \sum_{k=1}^{n} \left(e_{ik} a_k e_{jk}\right) = \sum_{y_k|t_i,\ y_k|tj} \left(\sum_{d|y_k,\ d\nmid y_u,\ y_u<y_k} g\,(d)\right) = \left(m_{ij}\right)_r.$$

**Theorem 9.** Let $T = \{t_1, t_2, \dots, t_n\}$ be a set of distinct positive integers, and $\overline{T} = \{y_1, y_2, \dots, y_m\}$ be the minimal gcd-closed set containing $T$. Then $(M^r) = A_r E^T$, where

$$a_{(ij)} = \begin{cases} \displaystyle\sum_{d|y_k,\ d\nmid y_u, y_u<y_k} g(d) & if\ y_j|t_i \\ 0 & otherwise \end{cases}$$

and

$$e_{(ij)} = \begin{cases} 1 & a_{ij} \neq 0 \\ 0 & otherwise \end{cases}$$

*Proof.* Since $A_r$ and $E$ are $n \times m$ matrices, then

$$(A_r E^T)_{ij} = \sum_{k=1}^{n} \left(a_{ik} e_{jk}\right) = \sum_{y_k|t_i,\ y_k|tj} \left(\sum_{d|y_k,\ d\nmid y_u, y_u<y_k} g\,(d)\right) = \left(m_{ij}\right)^r.$$

**Theorem 10.** Let $T = \{t_1, t_2, \dots, t_n\}$ be a non gcd-closed set of distinct positive integers, and let $\overline{T} = \{y_1, y_2, \dots, y_m\}$ be the minimal gcd-closed set containing $T$. Then $(M^r) = A_r A_r^T$, where

$$(A_r)_{(ij)} = \begin{cases} \sqrt{\displaystyle\sum_{d|y_k,\ d\nmid y_u, y_u<y_k} g(d)} & if\ y_j|t_i \\ 0 & otherwise \end{cases}.$$

*Proof.*

$$\begin{aligned}(A_r A_r^T)_{ij} &= \sum_{k=1}^{n} \left(a_{ik} a_{jk}\right) = \sum_{\substack{y_k|t_i \\ y_k|tj}} \sqrt{\sum_{d|y_k,\ d\nmid y_u, y_u<y_k} g\,(d)} \sqrt{\sum_{d|y_k,\ d\nmid y_u,\ y_u<y_k} g\,(d)} \\ &= \sum_{y_k|(t_i,t_j)} \left(\sum_{d|y_k,\ y_i<y_k,\ d\nmid y_u} g\,(d)\right) = \sum_{y_k|(t_i,t_j)} g\,(y_k) = \left(m_{ij}\right)^r.\end{aligned}$$

## Determinants

**Theorem 11.** Let $T = \{t_1, t_2, \dots, t_n\}$ be a non gcd-closed set of positive integers, and let $\overline{T} = \{y_1, y_2, \dots, y_m\}$ be the minimal gcd-closed set containing $T$ with $n < m$. If $E_{(k_1,k_2,\dots,k_m)_r}$ is the submatrix consisting of the $k_1^{th}, k_2^{th}, \dots, k_m^{th}$ columns of $E$ for some indices $k_i$ such that $1 \le k_1 < k_2 < \dots < k_m \le n$, then

$$\det(M^r) = \sum_{1\le k_1<k_2<\dots<k_m\le n} \left( \left(\det E_{(k_1,k_2,\dots,k_m)_r}\right)^2 \prod_{i=1}^{m} \left( \sum_{\substack{d|y_m \\ y_u<y_m \\ d\nmid y_u}} g\,(d) \right) \right)$$

*Proof.* Let $A = (a_{ij})$ and $E = (e_{ij})$ be its corresponding incidence matrix, where $a_{ij} = \left( \sum_{\substack{d|y_m \\ y_u<y_m \\ d\nmid y_u}} g\,(d) \right)$ if $y_j \mid t_i$ and 0 otherwise. But, $A$ is a diagonal matrix whose diagonal entries are $a_{ii} = \left( \sum_{\substack{d|y_m \\ y_u<y_m \\ d\nmid y_u}} g\,(d) \right)$ for all $1 \le i \le n$, so the $ij^{th}$ entry of $A$ may be written as $e_{ij} \left( \sum_{\substack{d|y_m \\ y_u<y_m \\ d\nmid y_u}} g\,(d) \right)$ and $(M^r) = EF^rE^T$.Define, for some indices $k_i$ such that $1 \le k_1 < k_2 < \dots < k_m \le n$, the matrices $A_{(k_1,k_2,\dots,k_m)}$ and $E_{(k_1,k_2,\dots,k_m)}$ to be the submatrices consisting of $k_1^{th}, k_2^{th}, \dots, k_m^{th}$ columns of $A$ and $E$ respectively, then $A_{(k_1,k_2,\dots,k_m)} = E_{(k_1,k_2,\dots,k_m)} D_r$, where $D_r$ is the $m \times m$ diagonal submatrix of $A_r$ whose diagonal elements are $d_{ii} = \left( \sum_{\substack{d|y_m \\ y_u<y_m \\ d\nmid y_u}} g\,(d) \right)$. Therefore, $\det\left(A_{(k_1,k_2,\dots,k_m)}\right) = \det\left(E_{(k_1,k_2,\dots,k_m)}\right)\left(\prod_{i=1}^{m} d_{ii}\right)$. Applying Cauchy-Binet formula, we get

$$\begin{aligned}
\det[M^r] &= \det A_r E^T \\
&= \sum_{1\le k_1<k_2<\dots<k_m\le n} \left( \left(\det A_{(k_1,k_2,\dots,k_m)}\right) \left(\det E_{(k_1,k_2,\dots,k_m)}\right)^T \right) \\
&= \sum_{1\le k_1<\dots<k_m\le n} \left( \det\left(E_{r(k_1,\dots,k_m)}\right) \left( \prod_{i=1}^{m} \left( \sum_{\substack{d|y_m \\ y_u<y_m \\ d\nmid y_u}} g\,(d) \right) \right) \left(\det E_{r(k_1,\dots,k_m)}\right)^T \right) \\
&= \sum_{1\le k_1<k_2<\dots<k_m\le n} \left( \prod_{i=1}^{m} \left( \sum_{\substack{d|y_m \\ y_u<y_m \\ d\nmid y_u}} g\,(d) \right) \left(\det E_{r(k_1,k_2,\dots,k_m)}\right)^2 \right).
\end{aligned}$$

**Example 2.** Consider the non gcd-closed set $T = \{2,4\}$ and its gcd-closure $\overline{T} = \{1,2,4\}$. Then, the $2^{nd}$ power Mersenne $GCD$ matrix defined on $T$ is:

$$M^2 = \begin{bmatrix} 17^2 & 17^2 \\ 17^2 & 65537^2 \end{bmatrix} = \begin{bmatrix} 289 & 289 \\ 289 & 4295098369 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} g(1) & 0 & 0 \\ 0 & g(2) & 0 \\ 0 & 0 & g(1)+g(4) \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 264 & 0 \\ 0 & 0 & 4295098105 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad E_{13} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } \quad E_{23} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

It is clear that $M^2 = EA_2E^T$. Applying Cauchy-Binet formula, we get

$$\begin{aligned} \det(M^2) &= \sum_{1\le k_1<k_2\le 3} \left( \det E_{(k_1,k_2)} \prod_{i=1}^{2} \big(g(t_i)\big) \right) \\ &= g(1)g(2)[\det(E_{12})]^2 + g(1)g(4)[\det(E_{13})]^2 + g(2)g(4)[\det(E_{23})]^2 \\ &= 0 + 107377452000 + 1133905893120 \\ &= 1241283345120 \end{aligned}$$

**Corollary 2.** Let $T = \{t_1, t_2, \dots, t_n\}$ be a gcd-closed set of distinct positive integers, then

$$\det[M^r] = \det(EA_rE^T) = \det(E)\det(A_r)\det(E^T) = \det(A_r) = \prod_{i=1}^{n} \left( \sum_{\substack{d|t_m \\ t_u<t_m \\ d\nmid t_u}} g\,(d) \right).$$

## Reciprocals and Inverses

**Theorem 12.** Let $T = \{t_1, t_2, \dots, t_n\}$ be a non gcd-closed set of distinct positive integers, and $\overline{T} = \{y_1, y_2, \dots, y_m\}$ be the minimal gcd-closed set containing $T$, then $(M^{-r}) = EA_{-r}E^T$, where

$$A_{-r} = diag\left( \sum_{d|y_1,\, d\nmid y_u,\, y_u<y_1} h\,(d), \sum_{d|y_2,\, d\nmid y_u,\, y_u<y_2} h\,(d), \dots, \sum_{d|y_m,\, d\nmid y_u,\, y_u<y_m} h\,(d) \right)$$

such that

$$h(n) = \sum_{d|n} \left( \frac{1}{(2^d - 1)} \right)^r \mu\left(\frac{n}{d}\right).$$

*Proof.* Let $\overline{T} = \{y_1, y_2, \dots, y_m\}$ be the minimal gcd-closed set containing $T$. Then,

$$\begin{aligned} (EA_{-r}E^T)_{ij} &= \sum_{k=1}^{n} \left(e_{ik}a_k e_{jk}\right) = \sum_{y_k|t_i,\, y_k|tj} \left( \sum_{d|y_k, y_i<y_k,\, d\nmid y_u} h\,(d) \right) \\ &= \sum_{y_k|(t_i,t_j)} h\,(y_k) = \left( \frac{1}{\left(2^{(t_i,t_j)} - 1\right)} \right)^r = (M^{-r})_{ij}. \end{aligned}$$

**Theorem 13.** Let $T = \{t_1, t_2, \dots, t_n\}$ be a gcd-closed set of positive integers, then the inverse of $(M^r)$ is $(M^r)^{-1}$ such that

$$(M^r)^{-1}_{ij} = \sum_{\substack{t_i|t_k \\ t_j|t_k}} \left( \frac{\mu\left(\frac{t_k}{t_i}\right)\mu\left(\frac{t_k}{t_j}\right)}{\sum_{d|y_k,\, d\nmid y_u,\, y_u<y_k} g\,(d)} \right)$$

*Proof.* Since $T$ is gcd-closed, then $E$ is an $n \times n$ square invertible matrix such that$E^{-1} = F^T$.Then,

$$\begin{aligned}(M^r)^{-1}_{ij} &= (EA_rE^T)^{-1}_{ij} = (E^{-1})^T(A_r)^{-1}(E^{-1}) = F^T(A_r)^{-1}F \\ &= \sum_{k=1}^{m} f_{ik}\frac{1}{f^r_{kk}}f_{kj} = \sum_{\substack{t_i|t_k \\ t_j|t_k}}\left(\frac{\mu\left(\frac{t_k}{t_i}\right)\mu\left(\frac{t_k}{t_j}\right)}{\sum_{d|y_k,\, d\nmid y_u,\, y_u<y_k} g\,(d)}\right)\end{aligned}$$

## Conclusion

In conclusion, this paper has presented a thorough investigation into the $n \times n$ Mersenne power GCD matrices defined on arbitrary sets of positive integers. By building upon prior research and utilizing a specialized form of the arithmetical function, we have explored the unique properties and behaviors of these matrices on both factor-closed and gcd-closed sets.

Our analysis has yielded valuable insights into the factorizations, determinants, reciprocals, and inverses of these Mersenne power GCD matrices. By elucidating their characteristics, we have contributed to the broader understanding of power GCD matrices and their applications in number theory and linear algebra. Furthermore, our findings not only expand upon existing knowledge but also pave the way for further exploration and refinement in this area of study. The versatility and significance of Mersenne power GCD matrices underscore their potential relevance in diverse mathematical contexts.

In essence, this paper underscores the importance of investigating specialized forms of power GCD matrices, such as the Mersenne variant, and highlights the rich interplay between number theory, algebra, and computational mathematics. Through our rigorous analysis, we have provided valuable insights that may inspire future research endeavors and contribute to the advancement of mathematical theory and practice.

## Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

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