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Analysis of Solutions for Nonlinear ψ-Caputo Fractional Differential Equations with Fractional Derivative Boundary Conditions in Banach Algebra

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Abstract: This article explores the solutions of nonlinear implicit ψ-Caputo fractional-order ordinary differential equations (NLIFDEs) with two-point fractional derivatives boundary conditions in Banach algebra. The research aims to establish the existence and uniqueness of solutions for this complex class of differential equations. Utilizing Banach's and Krasnoselskii's fixed point theorems, the study conducts a rigorous analysis of the solutions, ensuring their existence and uniqueness. This comprehensive investigation contributes to enhancing the understanding of the behavior of solutions of nonlinear fractional differentials within a challenging mathematical framework.

Keywords: Banach algebra, Nonlinear fractional differential equations, Derivative boundary conditions,

Introduction and Preliminaries

Fractional calculus extends traditional differentiation and integration to non-integer orders, offering a useful basis for modeling complex phenomena in various scientific and engineering disciplines. Esteemed mathematicians such as Almeida (2017) and Agarwal (2012) and Kiblas (2006) and Burton (1998) and Samko (1993) have made significant contributions to this field, expanding its scope.

One particular area of focus is nonlinear implicit fractional order differential equations (NLIFDEs) with fractional boundary conditions (FBCs), which find applications in mathematical physics, engineering sciences, and computational mathematics (Agrawal, 2009; Awad & AlKhezi, 2023; Awad, 2024;Benlabbes et al., 2015; Debazi & Hammouche, 2020)).

Almeida's work on ψ-fractional derivatives (Almeida, 2017), a generalization of Riemann-Liouville derivatives, introduces a Caputo-type regularization, explored extensively in some recent researches such as Awad and Kaddoura (2024) and Awad, (2023) and Awad et al. (2023), Kaddoura and Awad (2023). These researches have focused on the existence of positive solutions for fractional differential equations with boundary conditions, employing methodologies such as Banach's and Krasnoselskii's fixed point theorems.

This article aims to investigate the existence and uniqueness of solutions for the following nonlinear implicit ψ-Caputo fractional differential equations (NLIFDEs) with fractional boundary conditions within the domain of Banach Algebra:

$$\mathfrak{D}_{0^+}^{\alpha,\psi} y(t) = f\left(t, y(t), \mathfrak{D}_{0^+}^{\beta,\psi} y(t), \int_0^t k(t,s) \mathfrak{D}_{0^+}^{\alpha,\psi} y(s) ds\right), \tag{1}$$

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Subjected to the subsequent set of three integral boundary conditions involving fractional derivatives:

$$y(0) + \mathfrak{D}_{0^+}^{\alpha-1,\psi} y(T) = \sigma_1, \tag{2}$$

$$\mathfrak{D}_{0^+}^{\alpha-1,\psi} y(0) + \mathfrak{D}_{0^+}^{\alpha-2,\psi} y(T) = \sigma_2, \tag{3}$$

$$\mathfrak{D}_{0^+}^{\alpha-2,\psi} y(0) + \mathfrak{D}_{0^+}^{\alpha-3,\psi} y(T) = \sigma_3, \tag{4}$$

where $t \in J = [0,T], \mathfrak{D}_{0^+}^{\alpha,\psi}$ and $\mathfrak{D}_{0^+}^{\beta,\psi}$ denote the standard ψ-Caputo fractional derivatives of orders $\alpha \in (2,3]$ and $\beta \in (0,1]$, $\sigma_i \in R \ \forall \ (i = 1,2,3)$, $\psi(t)$ is an increasing function with $\psi'(t) \neq 0 \ \ \forall \ t \in J = [0,T]$, $f: J \times R^3 \to R$, and $k: J \times R \to R$ are continuous functions.

In the following, we present certain symbols, definitions, lemmas, and theorems that serve as foundational elements for our study. These essential concepts can be referenced in Almeida (2017), Burton (1998), Kiblas (2006), and (Samko, 1993), and related sources.

Definition 1.1: [1] Consider $\alpha > 0$, $n \in N$ such that $n = [\alpha] + 1$, and let $I = [a,b]$ represent an interval with $-\infty < a < t < b < +\infty$. Suppose $\psi, x \in C^n(I, \mathbb{R})$ are two functions, where ψ is increasing and $\psi'(t) \neq 0$ for all $t \in I$. In this context,

1) The left-sided ψ-Riemann-Liouville fractional integral of $x(t)$ of the fractional order α with respect to ψ is defined as:

$$\mathfrak{I}_{a^+}^{\alpha,\psi} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) \big(\psi(t) - \psi(s)\big)^{\alpha-1} x(s) ds,$$

2) where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} d\zeta$.

3) The left-sided ψ-Caputo fractional derivative of $x(t)$ of the fractional order α is defined as:

$$\begin{aligned} {}^c\mathfrak{D}_{a^+}^{\alpha,\psi} x(t) &= \mathfrak{I}_{a^+}^{n-\alpha,\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n x(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi'(s) \big(\psi(t) - \psi(s)\big)^{n-\alpha-1} x_\psi^{[n]}(s) ds, \end{aligned}$$

4) where $x_\psi^{[n]}(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n x(t)$.

Lemma 1.1: [17] If γ is a positive real number such that $\gamma > -1$ and $\gamma \neq \alpha - 1, \alpha - 2, \dots, \alpha - n$, then for $t \in I$,

$${}^c\mathfrak{D}_{0^+}^{\alpha,\psi} \big(\psi(t) - \psi(a)\big)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} \big(\psi(t) - \psi(a)\big)^{\gamma-\alpha}, \tag{5}$$

where ${}^c\mathfrak{D}_{0^+}^{\alpha,\psi} \big(\psi(t) - \psi(a)\big)^{\alpha-i} = 0$ for all $i = 1,2,3,\dots,n$.

Lemma 1.2. [1] If $\alpha > 0$, then the differential equation ${}^c\mathfrak{D}_{a^+}^{\alpha,\psi} x(t) = 0$ has a solution in $C(J,\mathbb{R}) \cap L_1(J,\mathbb{R})$ which is:

$$x(t) = c_1 \big(\psi(t) - \psi(0)\big)^{\alpha-1} + c_2 \big(\psi(t) - \psi(0)\big)^{\alpha-2} + \dots + c_n \big(\psi(t) - \psi(0)\big)^{\alpha-n},$$

where $c_i \in \mathbb{R}$ for all $i = 1,2,\dots,n$, and $n = [\alpha] + 1$.

Theorem 1.1. [15] (Banach's Fixed Point Theorem) Given a Banach space $(X, \|.\|)$, and a contraction mapping $\wp: X \to X$, there exists a unique fixed point $x \in X$ such that $\wp(x) = x$.

Theorem 1.2. [11] (Krasnselskii's fixed point theorem) Let $\mathcal{S}$ denote a closed, convex, and non-empty subset of a Banach space X. Suppose $\wp_1$ and $\wp_2$ are mappings from $\mathcal{S}$ to X satisfying the following conditions:

1) For any $u, v \in \mathcal{S}$, the sum $\wp_1 u + \wp_2 v$ belongs to $\mathcal{S}$.
2) The mapping $\wp_1$ is a contraction.
3) The mapping $\wp_2$ is continuous, and the range $\wp_2(\mathcal{S})$ is bounded.

Under these assumptions, there exists at least one element $u \in \mathcal{S}$ such that $\wp_1 u + \wp_2 u = u$.

Main Results

Definition 2.1. A function $y \in C(J, \mathbb{R})$ is considered as a solution if it satisfies both the nonlinear implicit fractional differential equation NLIFDE (1) and its associated boundary conditions (2)-(4).

Lemma 2.1. Suppose that $2 < \alpha \leq 3$, and let $f: J \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. A function $y(t) \in C(J, \mathbb{R})$ is considered as a solution to the nonlinear implicit fractional differential equation NLIFDE (**[1]**) if and only if it satisfies the the following fractional integral equation:

$$
\begin{aligned}
y(t) &= \frac{\left(\psi(t)-\psi(0)\right)^{\alpha-1}}{\Gamma(\alpha)}\left(\sigma_1 - \int_0^T \psi'(s)u(s)ds\right) \qquad (6)\\
&+ \frac{\left(\psi(t)-\psi(0)\right)^{\alpha-2}}{\Gamma(\alpha-1)}\left(\begin{array}{c}\sigma_2 + \varphi(T)\left(-\sigma_1 + \int_0^T \psi'(s)u(s)ds\right)\\ -\int_0^T \psi'(s)\left(\psi(T)-\psi(s)\right)u(s)\,ds.\end{array}\right)\\
&+ \frac{\left(\psi(t)-\psi(0)\right)^{\alpha-3}}{\Gamma(\alpha-2)}\left(\begin{array}{c}\sigma_3 - \frac{1}{2}\int_0^T \psi'(s)\left(\psi(T)-\psi(s)\right)^2 u(s)\,ds\\ -\frac{1}{2}\left(\sigma_1 - \int_0^T \psi'(s)u(s)ds\right)\left(\psi(T)-\psi(0)\right)^2\\ -\varphi(T)\left(\begin{array}{c}\sigma_2 + \varphi(T)\left(-\sigma_1 + \int_0^T \psi'(s)u(s)ds\right)\\ -\int_0^T \psi'(s)\left(\psi(T)-\psi(s)\right)u(s)\,ds\end{array}\right)\end{array}\right)\\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}u(s)ds,
\end{aligned}
$$

where $u(s)$ is the solution of the following fractional integral equation

$$u(t) = f\left(t, y(t), \mathfrak{I}_{0^+}^{\alpha-\beta,\psi}u(t), \int_0^t k(t,s)u(s)ds\right), \qquad (7)$$

Proof. Let $y(t)$ be a solution to the nonlinear implicit fractional differential equation NLIFDE (1). Define

$$u(t) = f\left(t, y(t), \mathfrak{D}_{0^+}^{\beta,\psi}y(t), \int_0^t k(t,s)\mathfrak{D}_{0^+}^{\alpha,\psi}y(s)ds\right).$$

It is clear that $\mathfrak{D}_{0^+}^{\beta,\psi}y(t) = \mathfrak{I}_{0^+}^{\alpha-\beta,\psi}\mathfrak{D}_{0^+}^{\alpha,\psi}y(t)$ for all $t \in J$. So, if $y(t)$ is a solution of equation (1), then for every $t \in J$, we have

$$\mathfrak{D}_{0^+}^{\alpha,\psi} y(t) = f\left(t, y(t), \mathfrak{I}_{0^+}^{\alpha-\beta,\psi} \mathfrak{D}_{0^+}^{\alpha,\psi} y(t), \int_0^t k\,(t,s) \mathfrak{D}_{0^+}^{\alpha,\psi} y(s) ds\right).$$

Let $\mathfrak{D}_{0^+}^{\beta,\psi} y(t) = u(t)$, then equation (1) becomes:

$$u(t) = f\left(t, y(t), \mathfrak{I}_{0^+}^{\alpha-\beta,\psi} u(t), \int_0^t k\,(t,s) u(s) ds\right).$$

Utilizing Lemma 1.2, we derive the expression:

$$\begin{aligned} y(t) = c_1\big(\psi(t)-\psi(0)\big)^{\alpha-1} &+ c_2\big(\psi(t)-\psi(0)\big)^{\alpha-2} \\ &+ c_3\big(\psi(t)-\psi(0)\big)^{\alpha-3} \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^T \psi'\,(s)\big(\psi(T)-\psi(s)\big)^2 u(s)\, ds \end{aligned} \tag{8}$$

Applying the boundary conditions (2)-(4), we obtain the following equations:

$$c_1\Gamma(\alpha) = \sigma_1 - \int_0^T \psi'\,(s) u(s) ds, \tag{9}$$

$$c_1\Gamma(\alpha)\big(1+\psi(T)-\psi(0)\big) + c_2\Gamma(\alpha-1) = \sigma_2 - \int_0^T \psi'\,(s)\big(\psi(T)-\psi(s)\big)u(s)\, ds, \tag{10}$$

and

$$\begin{aligned} &\frac{1}{2}c_1\Gamma(\alpha)\big(\psi(T)-\psi(0)\big)^2 + c_2\Gamma(\alpha-1)\big(1+\psi(T)-\psi(0)\big) + c_3\Gamma(\alpha-2) \\ &\qquad = \sigma_3 - \tfrac{1}{2}\int_0^T \psi'\,(s)\big(\psi(T)-\psi(s)\big)^2 u(s)\, ds. \end{aligned} \tag{11}$$

Solving equations (9), (10), and (11) for c_1, c_2, and c_3, we obtain:

$$c_1 = \frac{1}{\Gamma(\alpha)}\left(\sigma_1 - \int_0^T \psi'\,(s) u(s) ds\right),$$

$$c_2 = \frac{1}{\Gamma(\alpha-1)}\left(\varphi(T)\left(-\sigma_1 + \int_0^T \psi'\,(s) u(s) ds\right) + \sigma_2 - \int_0^T \psi'\,(s)\big(\psi(T)-\psi(s)\big)u(s)\, ds.\right),$$

and

$$c_3 = \frac{1}{\Gamma(\alpha-2)}\left(\begin{array}{c} \sigma_3 - \dfrac{1}{2}\displaystyle\int_0^T \psi'\,(s)\big(\psi(T)-\psi(s)\big)^2 u(s)\, ds \\ -\dfrac{1}{2}\left(\sigma_1 - \displaystyle\int_0^T \psi'\,(s) u(s) ds\right)\big(\psi(T)-\psi(0)\big)^2 \\ +\sigma_2 - \left(\begin{array}{c} \varphi(T)\left(-\sigma_1 + \displaystyle\int_0^T \psi'\,(s) u(s) ds\right) \\ -\displaystyle\int_0^T \psi'\,(s)\big(\psi(T)-\psi(s)\big)u(s)\, ds \end{array}\right)\varphi(T) \end{array}\right),$$

where $\varphi(T) = 1 + \psi(T) - \psi(0)$. Substituting these into (8), we obtain:

$$y(t) = \frac{\big(\psi(t)-\psi(0)\big)^{\alpha-1}}{\Gamma(\alpha)}\left(\sigma_1 - \int_0^T \psi'\,(s) u(s) ds\right)$$

$$+\frac{\left(\psi(t)-\psi(0)\right)^{\alpha-2}}{\Gamma(\alpha-1)}\begin{pmatrix}\sigma_2+\varphi(T)\left(-\sigma_1+\int_0^T\psi'(s)u(s)ds\right)\\-\int_0^T\psi'(s)\left(\psi(T)-\psi(s)\right)u(s)\,ds.\end{pmatrix}$$

$$+\frac{\left(\psi(t)-\psi(0)\right)^{\alpha-3}}{\Gamma(\alpha-2)}\begin{pmatrix}\sigma_3-\frac{1}{2}\int_0^T\psi'(s)\left(\psi(T)-\psi(s)\right)^2u(s)\,ds\\-\frac{1}{2}\left(\sigma_1-\int_0^T\psi'(s)u(s)ds\right)\left(\psi(T)-\psi(0)\right)^2\\-\varphi(T)\begin{pmatrix}\sigma_2+\varphi(T)\left(-\sigma_1+\int_0^T\psi'(s)u(s)ds\right)\\-\int_0^T\psi'(s)\left(\psi(T)-\psi(s)\right)u(s)\,ds\end{pmatrix}\end{pmatrix}$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}u(s)ds.$$

On the contrary, assume that $y(t)$ constitutes a solution to the nonlinear implicit fractional differential equation NLIFDE (1), and this solution can be expressed in the subsequent manner:

$$y(t)=\frac{\left(\psi(t)-\psi(0)\right)^{\alpha-1}}{\Gamma(\alpha)}\left(\sigma_1-\int_0^T\psi'(s)u(s)ds\right)$$

$$+\frac{\left(\psi(t)-\psi(0)\right)^{\alpha-2}}{\Gamma(\alpha-1)}\begin{pmatrix}\sigma_2+\varphi(T)\left(-\sigma_1+\int_0^T\psi'(s)u(s)ds\right)\\-\int_0^T\psi'(s)\left(\psi(T)-\psi(s)\right)u(s)\,ds.\end{pmatrix}$$

$$+\frac{\left(\psi(t)-\psi(0)\right)^{\alpha-3}}{\Gamma(\alpha-2)}\begin{pmatrix}\sigma_3-\frac{1}{2}\int_0^T\psi'(s)\left(\psi(T)-\psi(s)\right)^2u(s)\,ds\\-\frac{1}{2}\left(\sigma_1-\int_0^T\psi'(s)u(s)ds\right)\left(\psi(T)-\psi(0)\right)^2\\-\varphi(T)\begin{pmatrix}\sigma_2+\varphi(T)\left(-\sigma_1+\int_0^T\psi'(s)u(s)ds\right)\\-\int_0^T\psi'(s)\left(\psi(T)-\psi(s)\right)u(s)\,ds\end{pmatrix}\end{pmatrix}$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}u(s)ds.$$

Thus, we can infer that: $\mathfrak{D}_{0^+}^{\alpha,\psi}y(t)=u(t)$, with $y(0)+\mathfrak{D}_{0^+}^{\alpha-1,\psi}y(T)=\sigma_1$, $\mathfrak{D}_{0^+}^{\alpha-1,\psi}y(0)+\mathfrak{D}_{0^+}^{\alpha-2,\psi}y(T)=\sigma_2$, and $\mathfrak{D}_{0^+}^{\alpha-2,\psi}y(0)+\mathfrak{D}_{0^+}^{\alpha-3,\psi}y(T)=\sigma_3$. This implies that $u(t)$ indeed satisfies the conditions of problem (6). This concludes the proof.

Lemma 2.2. Consider the NLIFDE (1) under the following conditions:

(H_1) The nonlinear function $f: J\times\mathbb{R}^3\to\mathbb{R}$ exhibits continuity, and there exists $\lambda\in C(J,\mathbb{R}^+)$ such that:

$$|f(t,u_1,u_2,u_3)-f(t,v_1,v_2,v_3)|\leq\lambda(t)(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|),$$

for all $t\in J$, $u_i,v_i\in\mathbb{R}$, and $i=1,2,3$.

(H_2) The function $k(t,s)$ is continuous over $J\times J$, and there exists a positive constant K such that:

$$\max_{t,s\in[0,1]}|k(t,s)| = K.$$

Remark 1. Derived from Lemma, we extract that under the premise of (H_1), the inequality

$$|f(t,u_1,u_2,u_3)| - |f(t,0,0,0)| \leq |f(t,u_1,u_2,u_3) - f(t,0,0,0)| \leq \lambda(t)(|u_1| + |u_2| + |u_3|).$$

holds. Consequently, if $F = \sup_{t\in J}|f(t,0,0,0)|$, it follows that

$$|f(t,u_1,u_2,u_3)| \leq F + \lambda(t)(|u_1| + |u_2| + |u_3|)$$

Definition 2.2. Define the operator $\wp: C(J,\mathbb{R}) \to C(J,\mathbb{R})$ as follows:

$$\begin{aligned}
\wp\big(y(t)\big) = {} & \frac{\big(\psi(t)-\psi(0)\big)^{\alpha-1}}{\Gamma(\alpha)}\left(\sigma_1 - \int_0^T \psi'(s)u(s)ds\right) \\
& + \frac{\big(\psi(t)-\psi(0)\big)^{\alpha-2}}{\Gamma(\alpha-1)}\begin{pmatrix} \varphi(T)\left(-\sigma_1 + \int_0^T \psi'(s)u(s)ds\right) \\ +\sigma_2 - \int_0^T \psi'(s)\big(\psi(T)-\psi(s)\big)u(s)\,ds. \end{pmatrix} \\
& + \frac{\big(\psi(t)-\psi(0)\big)^{\alpha-3}}{\Gamma(\alpha-2)}\begin{pmatrix} \sigma_3 - \frac{1}{2}\int_0^T \psi'(s)\big(\psi(T)-\psi(s)\big)^2 u(s)\,ds \\ -\frac{1}{2}\left(\sigma_1 - \int_0^T \psi'(s)u(s)ds\right)\big(\psi(T)-\psi(0)\big)^2 \\ -\varphi(T)\begin{pmatrix} \sigma_2 + \varphi(T)\left(-\sigma_1 + \int_0^T \psi'(s)u(s)ds\right) \\ -\int_0^T \psi'(s)\big(\psi(T)-\psi(s)\big)u(s)\,ds \end{pmatrix} \end{pmatrix} \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\big(\psi(t)-\psi(s)\big)^{\alpha-1}u(s)ds.
\end{aligned}$$

where $u(s) \in C(J,\mathbb{R})$ satisfies the following implicit fractional equation:

$$u(t) = f\left(t, y(t), \Im_{0^+}^{\alpha-\beta,\psi}u(t), \int_0^t k(t,s)u(s)ds\right).$$

Existence of Solutions

In the following, we establish the existence of solutions for the Nonlinear Fractional Differential Equation NLIFDE defined by (1). Our approach centers on the application of Krasnoselskii's fixed point theorem.

Theorem 2.1. Suppose that assumptions (H_1) and (H_2) hold. If $\frac{\|\lambda\|\aleph}{\mathcal{M}} < 1$, where

$$\aleph = \begin{pmatrix} \frac{\big(\psi(T)-\psi(0)\big)^{\alpha}}{\Gamma(\alpha)} + \varphi(T)\frac{\big(\psi(T)-\psi(0)\big)^{\alpha-1}}{\Gamma(\alpha-1)} + \frac{\big(\psi(T)-\psi(0)\big)^{\alpha}}{2\Gamma(\alpha-1)} \\ + \frac{2\big(\psi(T)-\psi(0)\big)^{\alpha}}{3\Gamma(\alpha-2)} - \varphi^2(T)\frac{\big(\psi(T)-\psi(0)\big)^{\alpha-2}}{\Gamma(\alpha-2)} + \varphi(T)\frac{\big(\psi(T)-\psi(0)\big)^{\alpha-1}}{2\Gamma(\alpha-2)} \end{pmatrix}$$

and

$$\mathcal{M} = 1 - \|\lambda\|\left(\frac{\big(\psi(T)-\psi(0)\big)^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)} + KT\right),$$

then NLIFDE (1) has at least one solution in $C[0,1]$.

Proof. By transforming NLIFDE (1) into a problem involving fixed points, we introduce the operator $\wp: C(J,\mathbb{R}) \to C(J,\mathbb{R})$ as follows:

$$\wp\big(y(t)\big) = \wp_1\big(y(t)\big) + \wp_2\big(y(t)\big), \quad t \in [0,1],$$

Where

$$\begin{aligned}
\wp_1\big(y(t)\big) = {} & \frac{\big(\psi(t)-\psi(0)\big)^{\alpha-1}}{\Gamma(\alpha)}\left(\sigma_1 - \int_0^T \psi'(s)u(s)ds\right) \\
& + \frac{\big(\psi(t)-\psi(0)\big)^{\alpha-2}}{\Gamma(\alpha-1)}\left(\begin{array}{c} \sigma_2 + \varphi(T)\left(-\sigma_1 + \int_0^T \psi'(s)u(s)ds\right) \\ -\int_0^T \psi'(s)\big(\psi(T)-\psi(s)\big)u(s)\,ds. \end{array}\right) \\
& + \frac{\big(\psi(t)-\psi(0)\big)^{\alpha-3}}{\Gamma(\alpha-2)}\left(\begin{array}{c} \sigma_3 - \frac{1}{2}\int_0^T \psi'(s)\big(\psi(T)-\psi(s)\big)^2 u(s)\,ds \\ -\frac{1}{2}\left(\sigma_1 - \int_0^T \psi'(s)u(s)ds\right)\big(\psi(T)-\psi(0)\big)^2 \\ -\varphi(T)\left(\begin{array}{c} \sigma_2 + \varphi(T)\left(-\sigma_1 + \int_0^T \psi'(s)u(s)ds\right) \\ -\int_0^T \psi'(s)\big(\psi(T)-\psi(s)\big)u(s)\,ds \end{array}\right) \end{array}\right),
\end{aligned}$$

and

$$\wp_2\big(y(t)\big) = \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\big(\psi(t)-\psi(s)\big)^{\alpha-1}u(s)ds,$$

with

$$u(t) = f\left(t,\ y(t),\ \mathfrak{I}_{0^+}^{\alpha-\beta,\psi}u(t), \int_0^t k(t,s)u(s)ds\right).$$

Consider $B_\varrho = \{y \in C(J,\mathbb{R}) : \|y\| \le \varrho\}$ as a closed subset of $C[0,1]$, where ϱ represents a positive constant satisfying $\varrho \ge \frac{\Re}{1-\aleph}$. Here, $\Re$ and $\aleph$ are real numbers. It is evident that B_ϱ constitutes a Banach space equipped with a metric in $C[0,T]$. The proof can be outlined in three distinct phases.

Step 1: $\wp_1 y_1 + \wp_2 y_2 \in B_\varrho$ holds true for all $y_1, y_2 \in B_\varrho$.

Consider $y_1, y_2 \in B_\varrho$ and $t \in J$. We obtain:

$$\begin{aligned}
\big|\wp_1\big(y_1(t)\big) + \wp_2\big(y_2(t)\big)\big| & \le \big|\wp_1\big(y_1(t)\big)\big| + \big|\wp_2\big(y_2(t)\big)\big| \\
& \le \frac{|\psi(t)-\psi(0)|^{\alpha-1}}{\Gamma(\alpha)}\left(\sigma_1 + \int_0^T \psi'(s)|u_1(s)|ds\right) \\
& + \frac{|\psi(t)-\psi(0)|^{\alpha-2}}{\Gamma(\alpha-1)}\left(\begin{array}{c} \sigma_2 + |\varphi(T)|\left(\sigma_1 + \int_0^T \psi'(s)|u_1(s)|ds\right) \\ +\int_0^T \psi'(s)\big(\psi(T)-\psi(s)\big)|u_1(s)|\,ds. \end{array}\right) \\
& + \frac{|\psi(t)-\psi(0)|^{\alpha-3}}{\Gamma(\alpha-2)}\left(\begin{array}{c} \sigma_3 + \frac{1}{2}\int_0^T \psi'(s)\big(\psi(T)-\psi(s)\big)^2 |u_1(s)|\,ds \\ +\frac{1}{2}\left(\sigma_1 + \int_0^T \psi'(s)|u_1(s)|ds\right)\big(\psi(T)-\psi(0)\big)^2 \\ +|\varphi(T)|\left(\begin{array}{c} \sigma_2 + |\varphi(T)|\left(\sigma_1 + \int_0^T \psi'(s)|u_1(s)|ds\right) \\ +\int_0^T \psi'(s)\big(\psi(T)-\psi(s)\big)|u_1(s)|\,ds \end{array}\right) \end{array}\right) \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\big(\psi(t)-\psi(s)\big)^{\alpha-1}|u_2(s)|ds.
\end{aligned}$$

Using Lemma 2.2 and the aforementioned remark, if we consider the supremum for $t \in [0, T]$, then

$$
\begin{aligned}
\|u\| &= \left| f\left(t,\ y(t),\ \mathfrak{J}_{0^+}^{\alpha-\beta,\psi} u(t), \int_0^t k\,(t,s)u(s)ds\right) \right| \\
&\leq \|\lambda\| \left(|y(t)| + \left|\mathfrak{J}_{0^+}^{\alpha-\beta,\psi} u(t)\right| + \left|\int_0^t k\,(t,s)u(s)ds\right| \right) + F \\
&\leq \|\lambda\| \left(\|y\| + \left(\frac{\left(\psi(T)-\psi(0)\right)^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)} + KT \right) \|u\| \right) + F,
\end{aligned}
$$

where $F = \sup_{t\in J} |f(t, 0,0,0)|$.

Hence,

$$
\|u\| \leq \frac{\|\lambda\|\|y\| + F}{\mathcal{M}},
$$

where $\mathcal{M} = 1 - \|\lambda\| \left(\frac{\left(\psi(T)-\psi(0)\right)^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)} + KT \right)$.

Thus, for each $t \in [0, T]$ we have

$$
\begin{aligned}
&\left|\wp_1\big(y_1(t)\big) + \wp_2\big(y_2(t)\big)\right| \leq \left|\wp_1\big(y_1(t)\big)\right| + \left|\wp_2\big(y_2(t)\big)\right| \\
&\leq \frac{|\psi(t)-\psi(0)|^{\alpha-1}}{\Gamma(\alpha)} \left(\sigma_1 + \left(\frac{\|\lambda\|\|y_1\| + F}{\mathcal{M}}\right) \big(\psi(T)-\psi(0)\big) \right) \\
&+ \frac{|\psi(t)-\psi(0)|^{\alpha-2}}{\Gamma(\alpha-1)} \left(\sigma_2 + |\varphi(T)| \left(\sigma_1 + \left(\frac{\|\lambda\|\|y_1\| + F}{\mathcal{M}}\right) \big(\psi(T)-\psi(0)\big) \right) + \frac{\left(\frac{\|\lambda\|\|y_1\| + F}{\mathcal{M}}\right) \big(\psi(T)-\psi(0)\big)^2}{2} \right) \\
&+ \frac{|\psi(t)-\psi(0)|^{\alpha-3}}{\Gamma(\alpha-2)} \left(\begin{array}{c} \sigma_3 + \frac{\left(\frac{\|\lambda\|\|y_1\| + F}{\mathcal{M}}\right) \big(\psi(T)-\psi(0)\big)^3}{6} + \left(\frac{\|\lambda\|\|y_1\| + F}{\mathcal{M}}\right) \varphi^2(T) \big(\psi(T)-\psi(0)\big) \\ + \sigma_1 \varphi^2(T) + \sigma_1 \frac{\big(\psi(T)-\psi(0)\big)^2}{2} + \frac{\left(\frac{\|\lambda\|\|y_1\| + F}{\mathcal{M}}\right) \big(\psi(T)-\psi(0)\big)^3}{2} \\ + \sigma_2 \varphi(T) + \frac{\left(\frac{\|\lambda\|\|y_1\| + F}{\mathcal{M}}\right) \varphi(T) \big(\psi(T)-\psi(0)\big)^2}{2} \end{array} \right) \\
&+ \frac{|\psi(t)-\psi(0)|^{\alpha}}{\Gamma(\alpha+1)} \left(\frac{\|\lambda\|\|y_2\| + F}{\mathcal{M}} \right).
\end{aligned}
$$

Taking supremum over $t \in [0, T]$, we have

$$
\|\wp_1 y_1(t) + \wp_2 y_2(t)\| \leq \varrho,
$$

for $\varrho \geq \frac{\Re}{1-\aleph}$, where

$$
\Re = \left(\sigma_1 \Upsilon_1 + \sigma_2 \Upsilon_2 + \sigma_3 \Upsilon_3 + (\|\lambda\| + F)\Upsilon_4\right)
$$

such that

$$\begin{aligned}\Upsilon_1 &= \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha-2)}\right)\big(\psi(T)-\psi(0)\big)^{\alpha-1} + \varphi(T)\frac{\big(\psi(T)-\psi(0)\big)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\qquad + \varphi^2(T)\frac{\big(\psi(T)-\psi(0)\big)^{\alpha-3}}{\Gamma(\alpha-2)},\end{aligned}$$

$$\Upsilon_2 = \frac{\big(\psi(T)-\psi(0)\big)^{\alpha-2}}{\Gamma(\alpha-1)} + \varphi(T)\frac{\big(\psi(T)-\psi(0)\big)^{\alpha-3}}{\Gamma(\alpha-2)},$$

$$\Upsilon_3 = \frac{\big(\psi(T)-\psi(0)\big)^{\alpha-3}}{\Gamma(\alpha-2)},$$

$$\begin{aligned}\Upsilon_4 &= \left(\frac{1}{|\Gamma(\alpha+1)|} + \frac{1}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha-1)} + \frac{2}{3\Gamma(\alpha-2)}\right)\big(\psi(T)-\psi(0)\big)^{\alpha} \\ &\qquad + \varphi(T)\left(\frac{1}{\Gamma(\alpha-1)} + \frac{1}{2\Gamma(\alpha-2)}\right)\big(\psi(T)-\psi(0)\big)^{\alpha-1} \\ &\qquad + \varphi^2(T)\frac{\big(\psi(T)-\psi(0)\big)^{\alpha-2}}{\Gamma(\alpha-2)},\end{aligned}$$

and

$$\aleph = \left(\begin{array}{c}\dfrac{\big(\psi(T)-\psi(0)\big)^{\alpha}}{\Gamma(\alpha)} + \varphi(T)\dfrac{\big(\psi(T)-\psi(0)\big)^{\alpha-1}}{\Gamma(\alpha-1)} + \dfrac{\big(\psi(T)-\psi(0)\big)^{\alpha}}{2\Gamma(\alpha-1)} \\ + \dfrac{2\big(\psi(T)-\psi(0)\big)^{\alpha}}{3\Gamma(\alpha-2)} - \varphi^2(T)\dfrac{\big(\psi(T)-\psi(0)\big)^{\alpha-2}}{\Gamma(\alpha-2)} + \varphi(T)\dfrac{\big(\psi(T)-\psi(0)\big)^{\alpha-1}}{2\Gamma(\alpha-2)}\end{array}\right)$$

This proves that $\wp_1 y_1(t) + \wp_2 y_2(t) \in B_\varrho$ for every $y_1, y_2 \in B_\varrho$.

Step 2: The operator $\wp_1$ serves as a contraction mapping on B_ϱ.

It is clear that

$$\begin{aligned}\|u_1 - u_2\| &= \left| f\left(t,\ y_1(t),\ \Im_{0^+}^{\alpha-\beta,\psi} u_1(t), \int_0^t k\,(t,s) u_1(s) ds\right) - f\left(t,\ y_2(t),\ \Im_{0^+}^{\alpha-\beta,\psi} u_2(t), \int_0^t k\,(t,s) u_2(s) ds\right)\right| \\ &\le \lambda(t)\left(|y_1(t) - y_2(t)| + \int_0^t \frac{\psi'(s)\big(\psi(t)-\psi(s)\big)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |u_1(t) - u_2(t)| ds \right. \\ &\left. \quad + \int_0^t |k\,(t,s)||u_1(t) - u_2(t)| ds\right)\end{aligned}$$

Taking supremum for all $t \in I$, we get

$$\|u_1 - u_2\| \le \|\lambda\| \left[\|y_1 - y_2\| + \left(\frac{\big(\psi(T)-\psi(0)\big)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + KT\right)\|u_1 - u_2\|\right].$$

Thus,

$$\|u_1 - u_2\| \le \frac{\|\lambda\|}{\mathcal{M}} \|y_1 - y_2\|.$$

This implies that

$$
\begin{aligned}
&\left|\wp_1(y_1(t)) - \wp_1(y_2(t))\right| \\
&\qquad = \frac{\left(\psi(t)-\psi(0)\right)^{\alpha-1}}{\Gamma(\alpha)} \int_0^T \psi'(s)|u_2(s)-u_1(s)|ds \\
&\qquad + \frac{\left(\psi(t)-\psi(0)\right)^{\alpha-2}}{\Gamma(\alpha-1)} \begin{pmatrix} \varphi(T)\left(\int_0^T \psi'(s)|u_1(s)-u_2(s)|ds\right) \\ +\int_0^T \psi'(s)\left(\psi(T)-\psi(s)\right)|u_2(s)-u_1(s)|\, ds \end{pmatrix} \\
&\qquad + \frac{\left(\psi(t)-\psi(0)\right)^{\alpha-3}}{\Gamma(\alpha-2)} \begin{pmatrix} +\frac{1}{2}\int_0^T \psi'(s)\left(\psi(T)-\psi(s)\right)^2|u_2(s)-u_1(s)|\, ds \\ +\frac{1}{2}\int_0^T \psi'(s)|u_1(s)-u_2(s)|ds\left(\psi(T)-\psi(0)\right)^2 \\ -\varphi(T)\begin{pmatrix} \varphi(T)\int_0^T \psi'(s)|u_1(s)-u_2(s)|ds \\ -\int_0^T \psi'(s)\left(\psi(T)-\psi(s)\right)|u_2(s)-u_1(s)|\, ds \end{pmatrix} \end{pmatrix}.
\end{aligned}
$$

Taking supremum over $t \in [0,T]$, we get

$$
\begin{aligned}
&\|\wp_1(y_1) - \wp_1(y_2)\| \\
&\qquad \le \frac{\left(\psi(T)-\psi(0)\right)^{\alpha}}{\Gamma(\alpha)} \|u_1-u_2\| \\
&\qquad + \frac{\left(\psi(T)-\psi(0)\right)^{\alpha-2}}{\Gamma(\alpha-1)} \left(\varphi(T)\left(\psi(T)-\psi(0)\right) + \frac{\left(\psi(T)-\psi(0)\right)^2}{2}\right)\|u_1-u_2\| \\
&\qquad + \frac{\left(\psi(T)-\psi(0)\right)^{\alpha-3}}{\Gamma(\alpha-2)} \begin{pmatrix} +\frac{\left(\psi(T)-\psi(0)\right)^3}{6} + \frac{\left(\psi(T)-\psi(0)\right)^3}{2} \\ -\varphi^2(T)\left(\psi(T)-\psi(0)\right) + \varphi(T)\frac{\left(\psi(T)-\psi(0)\right)^2}{2} \end{pmatrix}\|u_1-u_2\| \\
&\qquad \le \begin{pmatrix} \frac{\left(\psi(T)-\psi(0)\right)^{\alpha}}{\Gamma(\alpha)} + \varphi(T)\frac{\left(\psi(T)-\psi(0)\right)^{\alpha-1}}{\Gamma(\alpha-1)} + \frac{\left(\psi(T)-\psi(0)\right)^{\alpha}}{2\Gamma(\alpha-1)} \\ +\frac{2\left(\psi(T)-\psi(0)\right)^{\alpha}}{3\Gamma(\alpha-2)} - \varphi^2(T)\frac{\left(\psi(T)-\psi(0)\right)^{\alpha-2}}{\Gamma(\alpha-2)} + \varphi(T)\frac{\left(\psi(T)-\psi(0)\right)^{\alpha-1}}{2\Gamma(\alpha-2)} \end{pmatrix}\|u_1 \\
&\qquad - u_2\| \le \frac{\|\lambda\|\aleph}{\mathcal{M}}\|y_1-y_2\|.
\end{aligned}
$$

Thus, it is clear that the operator$\wp_1$ is a contraction mapping with a contraction coefficient $\frac{\|\lambda\|\aleph}{\mathcal{M}} < 1$.

Step 3: To establish the continuity and compactness of the operator $\wp_2$ on B_ϱ, we initially establish its continuity. Let $\{y_n\}_{n\in N}$ be a sequence in B_ϱ that converges to $y \in B_\varrho$ as n tends to infinity. Our objective is to demonstrate that $\|\wp_2 y_n - \wp_1 y\|$ tends to zero as n tends to infinity. Subsequently, for $t \in [0,T]$, we have:

$$
|\wp_2 y_n - \wp_2 y| \le \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}|u_n(s)-u(s)|ds,
$$

Where

$$
u_n(t) = f\left(t,\ y_n(t),\ \mathfrak{D}_{0^+}^{\beta,\psi} y_n(t), \int_0^t k(t,s)\mathfrak{D}_{0^+}^{\alpha,\psi} y_n(s)ds\right),
$$

and

$$
u(t) = f\left(t, y(t), \mathfrak{D}_{0^+}^{\beta} y(t), \int_0^t k(t,s)\mathfrak{D}_{0^+}^{\alpha,\psi} y(s)ds\right)
$$

are two continuous functions defined over $[0,T]$ such that

$$\begin{aligned}&|u_n(t)-u(t)|\\&=\left|f\left(t,y_n(t),\mathfrak{D}_{0^+}^{\beta}y_n(t),\int_0^t k(t,s)\mathfrak{D}_{0^+}^{\alpha,\psi}y_n(s)ds\right)-f\left(t,y(t),\mathfrak{D}_{0^+}^{\beta}y(t),\int_0^t k(t,s)\mathfrak{D}_{0^+}^{\alpha,\psi}y(s)ds\right)\right|,\\&\leq|\lambda(t)|\left(|y_n(t)-y(t)|+\left|\mathfrak{D}_{0^+}^{\beta}y_n(t)-\mathfrak{D}_{0^+}^{\beta}y(t)\right|+\int_0^t|k(t,s)|\left|\mathfrak{D}_{0^+}^{\alpha,\psi}y_n(s)-\mathfrak{D}_{0^+}^{\alpha,\psi}y(s)\right|ds\right)|),\\&\leq\frac{\|\lambda\|}{\mathcal{M}}\|y_n-y\|.\end{aligned}$$

Since $y_n\to y$, then we get $u_n(t)\to u(t)$ as $n\to\infty$ for each $t\in[0,T]$. And let $\varepsilon>0$ be such that, for each $t\in[0,T]$, we have $|u_n(t)|\leq\varepsilon/2$ and $|u(t)|\leq\varepsilon/2$ which implies that $|u_n(s)-u(s)|\leq(|u_n(s)|+|u(s)|)\leq\varepsilon$ for each $t\in[0,T]$. Applying Lebesgue Dominated Convergence Theorem, it implies that $\|\wp_2 y_n-\wp_2 y\|\to 0$ as $n\to\infty$. Consequently, operator $\wp_2$ is continuous. In addition, we have

$$\|\wp_2 y\|\leq\frac{1}{|\Gamma(\alpha+1)|}\left(\frac{\|\lambda\|\|y_1\|+F}{\mathcal{M}}\right)\leq\varrho$$

due to definitions of ϱ. This proves that $\wp_2$ is uniformly bounded on B_ϱ.

Ultimately, we demonstrate that the mapping $\wp_2$ transforms bounded sets into equicontinuous sets within $C(J,R)$, specifically ensuring the equicontinuity of B_ϱ.

Assume that $\forall\ \epsilon>0$, $\exists\ \delta>0$ and $t_1,t_2\in J,\ t_1<t_2,\ \ |t_2-t_1|<\delta$. Then, we have

$$\begin{aligned}|\wp_2 y(t_2)-\wp_2 y(t_1)|\quad&\leq\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)\left(\left(\psi(t_2)-\psi(s)\right)^{\alpha-1}-\left(\psi(t_1)-\psi(s)\right)^{\alpha-1}\right)|u(s)|ds,\\&\leq\left(\frac{\|\lambda\|\|y\|}{\mathcal{M}}\right)\frac{(\psi(t_2)^{\alpha}-\psi(t_1)^{\alpha})}{\alpha\Gamma(\alpha)}.\end{aligned}$$

As t_1 approaches t_2, the expression on the right-hand side of the aforementioned inequality becomes independent of y and approaches zero. Thus,

$$|\wp_2 y(t_2)-\wp_2 y(t_1)|\to 0,\ \ \forall\ |t_2-t_1|\to 0.$$

Thus, if $\wp$ is uniformly continuous on B_ϱ, where $\wp$ represents a compact operator, the Arzela-Ascoli theorem guarantees that $\wp: C([0,T],R)\to C([0,T],R)$ is both continuous and compact. Consequently, all the conditions of Krasnoselskii's fixed point theorem are satisfied, and the operator $\wp=\wp_1+\wp_2$ possesses a fixed point $y(t)\in C[0,T]$ on B_ϱ satisfying the boundary conditions in (1). As a result, $y(t)$ serves as a solution of the NLIFDE (1). This concludes the proof. □

Uniqueness of Solutions

Next, we ascertain the unique solutions to the nonlinear fractional differential equation NLIFDE (1). This exploration into uniqueness adds a valuable dimension to our understanding of the solutions in the context of our studied equation.

Theorem 2.3. If assumptions (H_1) and (H_2) hold, and if

$$\left(\aleph+\frac{\left(\psi(T)-\psi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}\right)\frac{\|\lambda\|}{\mathcal{M}}<1,$$

then operator $\wp: C(J,R)\to C(J,R)$ presented in Definition 2.2 is a contraction.

Proof. Assuming that conditions (H_1) and (H_2) are satisfied, let's examine the continuous functions $y_1(t)$ and $y_2(t)$ belonging to $C(J,R)$. In this context, for any $t \in J$, the following applies:

$$\left|\wp(y_1(t)) - \wp(y_2(t))\right|$$

$$\leq \left|\wp_1(y_1(t)) - \wp_1(y_2(t)) + \wp_2(y_1(t)) - \wp_2(y_2(t))\right|$$

$$\leq \left|\wp_1(y_1(t)) - \wp_1(y_2(t))\right| + \left|\wp_2(y_1(t)) - \wp_2(y_2(t))\right|$$

$$\leq \frac{(\psi(t)-\psi(0))^{\alpha-1}}{\Gamma(\alpha)} \int_0^T \psi'(s)|u_2(s)-u_1(s)|ds$$

$$+ \frac{(\psi(t)-\psi(0))^{\alpha-2}}{\Gamma(\alpha-1)} \begin{pmatrix} \varphi(T)\left(\int_0^T \psi'(s)|u_1(s)-u_2(s)|ds\right) \\ + \int_0^T \psi'(s)(\psi(T)-\psi(s))|u_2(s)-u_1(s)|\,ds. \end{pmatrix}$$

$$+ \frac{(\psi(t)-\psi(0))^{\alpha-3}}{\Gamma(\alpha-2)} \begin{pmatrix} +\frac{1}{2}\int_0^T \psi'(s)(\psi(T)-\psi(s))^2|u_2(s)-u_1(s)|\,ds \\ +\frac{1}{2}\int_0^T \psi'(s)|u_1(s)-u_2(s)|ds(\psi(T)-\psi(0))^2 \\ -\varphi(T)\begin{pmatrix} \varphi(T)\int_0^T \psi'(s)|u_1(s)-u_2(s)|ds \\ -\int_0^T \psi'(s)(\psi(T)-\psi(s))|u_2(s)-u_1(s)|\,ds \end{pmatrix} \end{pmatrix}$$

$$+ \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|u_1(s)-u_2(s)|ds$$

$$\leq \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha)}\|u_1-u_2\| + \frac{(\psi(T)-\psi(0))^{\alpha-2}}{\Gamma(\alpha-1)}\left(\varphi(T)(\psi(T)-\psi(0)) + \frac{(\psi(T)-\psi(0))^2}{2}\right)\|u_1-u_2\|$$

$$+ \frac{(\psi(T)-\psi(0))^{\alpha-3}}{\Gamma(\alpha-2)} \begin{pmatrix} +\frac{(\psi(T)-\psi(0))^3}{6} + \frac{(\psi(T)-\psi(0))^3}{2} \\ -\varphi^2(T)(\psi(T)-\psi(0)) + \varphi(T)\frac{(\psi(T)-\psi(0))^2}{2} \end{pmatrix}\|u_1-u_2\|$$

$$+ \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\|u_1-u_2\|$$

Taking supremum for all $t \in T$, we have

$$\left|\wp(y_1(t)) - \wp(y_2(t))\right| \leq \begin{pmatrix} \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha)} + \varphi(T)\frac{(\psi(T)-\psi(0))^{\alpha-1}}{\Gamma(\alpha-1)} \\ + \frac{(\psi(T)-\psi(0))^{\alpha}}{2\Gamma(\alpha-1)} + \frac{2(\psi(T)-\psi(0))^{\alpha}}{3\Gamma(\alpha-2)} \\ -\varphi^2(T)\frac{(\psi(T)-\psi(0))^{\alpha-2}}{\Gamma(\alpha-2)} + \varphi(T)\frac{(\psi(T)-\psi(0))^{\alpha-1}}{2\Gamma(\alpha-2)} \end{pmatrix}\|u_1-u_2\|$$

$$\leq \begin{pmatrix} \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha)} + \varphi(T)\frac{(\psi(T)-\psi(0))^{\alpha-1}}{\Gamma(\alpha-1)} \\ + \frac{(\psi(T)-\psi(0))^{\alpha}}{2\Gamma(\alpha-1)} + \frac{2(\psi(T)-\psi(0))^{\alpha}}{3\Gamma(\alpha-2)} \\ -\varphi^2(T)\frac{(\psi(T)-\psi(0))^{\alpha-2}}{\Gamma(\alpha-2)} + \varphi(T)\frac{(\psi(T)-\psi(0))^{\alpha-1}}{2\Gamma(\alpha-2)} \end{pmatrix}\frac{\|\lambda\|}{\mathcal{M}}\|y_1-y_2\|$$

$$\leq \left(\aleph + \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right)\frac{\|\lambda\|}{\mathcal{M}}\|y_1-y_2\|.$$

Thus,

$$\|\wp(y_1) - \wp(y_2)\| \leq \Delta\|y_1 - y_2\|,$$

where $\Delta = \left(\aleph + \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right)\frac{\|\lambda\|}{\mathcal{M}}$. By taking $\Delta < 1$, we obtain that the operator $\wp$ is contraction.

Applying Krasnselskii's fixed point theorem, we deduce that the nonlinear fractional differential equation NLIFDE (1) has at least one solution. □

Theorem 2.2. If assumptions (H_1) and (H_2) hold, and if

$$\left(\aleph + \frac{(\psi(T) - \psi(0))^{\alpha}}{\Gamma(\alpha + 1)}\right)\frac{\|\lambda\|}{\mathcal{M}} < 1,$$

then NLIFDE (1) has a unique solution on $J = [0, T]$.

Proof. The existence of at least one solution for NLIFDE (1) has been established in Theorem 2.1. Furthermore, Lemma 2.3 demonstrates that the operator $\wp$ exhibits contraction properties. Consequently, through Banach's fixed point theorem, we conclude that the operator $\wp$ possesses a single fixed point, which corresponds to a unique solution of the NLIFDE (1) over the interval $J = [0, T]$. Thus, the proof is now fully accomplished. □

Numerical Example

Consider the following NLIFDE:

$$\begin{cases} \mathfrak{D}^{\frac{11}{5},2t^3+1}y(t) = \dfrac{\sqrt{2t+1}}{59e^{2t+1}}\left[\dfrac{11 + y(t) + \mathfrak{D}^{\frac{3}{5},2t^3+1}y(t) + \int_0^1 e^{3(t-s)}\ \mathfrak{D}^{\frac{11}{5},2t^3+1}y(s)ds}{1 + y(t) + \mathfrak{D}^{\frac{3}{5},2t^3+1}y(t) + 2\int_0^1 e^{3(t-s)}\ \mathfrak{D}^{\frac{11}{5},2t^3+1}y(s)ds}\right] \text{ for all } t \in [0,1], \\ y(0) + \mathfrak{D}_{0^+}^{\frac{6}{5},2t^3+1}y(1) = 1.5, \\ \mathfrak{D}_{0^+}^{\frac{6}{5},2t^3+1}y(0) + \mathfrak{D}_{0^+}^{\frac{1}{5},2t^3+1}y(1) = 2.5, \\ \mathfrak{D}_{0^+}^{\frac{1}{5},2t^3+1}y(0) + \mathfrak{D}_{0^+}^{\frac{-4}{5},2t^3+1}y(1) = 3.5. \end{cases} \tag{13}$$

In this problem, we have $\alpha = \frac{11}{5}$, $\beta = \frac{3}{5}$, $\psi(t) = 2t^3 + 1$ which is an increasing function on $[0,1]$, $K(t,s) = e^{3(t-s)}$, $\sigma_1 = 1.5$,$\sigma_2 = 2.5$, $\sigma_3 = 3.5$.

It is clear that the assumptions (H_1) and (H_2) are satisfied, and f is a mutually continuous function such that for any u, v, $w \in R$, and $t \in [0,1]$ we have

$$|f(t, u, v, w)| = \frac{\sqrt{2t + 1}}{59e^{2t+1}}(11 + |u| + |v| + |w|),$$

with $\lambda(t) = \frac{\sqrt{2t+1}}{59e^{2t+1}}$, $F = \frac{11}{59e}$, $\|\lambda\| = \frac{1}{59e}$, and $K = e^3$.

It is clear from Theorem 2.1 that the nonlinear fractional integral differential equation (NLIFDE) (13) possesses at least one solution within the interval [0,1] since

$$\frac{\|\lambda\|\aleph}{\mathcal{M}} = \left(\frac{1}{59e}\right)\frac{17.8487}{0.86154} \approx 0.129177 < 1,$$

Moreover, by employing Theorem 2.2,the solution is unique since

$$\left(\aleph + \frac{\left(\psi(T) - \psi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}\right)\frac{\|\lambda\|}{\mathcal{M}} \approx 0.142895 < 1.$$

Conclusion

In conclusion, this article has delved into the intricate realm of nonlinear implicit ψ-Caputo fractional differential equations (NLIFDEs) with two-point fractional derivatives boundary conditions in Banach algebra. Through the application of Banach's and Krasnoselskii's fixed point theorems, we have rigorously established the existence and uniqueness of solutions within this complex mathematical framework.

Our investigation sheds light on the behavior of solutions for NLIFDEs, providing valuable insights into their dynamics and properties. By addressing this challenging class of differential equations, we contribute to advancing the understanding of nonlinear fractional differentials and their applications across diverse fields, including mathematical physics, engineering sciences, and computational mathematics.

Future research endeavors could extend this work by exploring additional classes of NLIFDEs with different types of boundary conditions or investigating the stability and numerical methods for solving such equations. By continuing to push the boundaries of knowledge in fractional calculus, we can unlock new avenues for modeling and analyzing complex phenomena in various scientific and engineering disciplines.

Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

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