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General Upper Bounds for the Numerical Radii of Hilbert Space Operators

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Abstract: We present a collection upper bounds for the numerical radii of a certain 2×2 operator matrices. We use these bounds to improve on some known numerical radius inequalities for powers of Hilbert space operators. In particular, we show that if A is a bounded linear operator on a complex Hilbert space, then $w^{2r}(A)\le \frac{1+\alpha}{8}\||A|^{2r}+|A^*|^{2r}\|+\frac{1+\alpha}{4}w(|A|^r|A^*|^r)+\frac{1-\alpha}{2}w^r(A^2)$ for every $r\ge 1$ and $\alpha\in[0,1]$. This substantially improves on the existing inequality $w^{2r}(A)\le\frac{1}{2}\||A|^{2r}+|A^*|^{2r}\|$. Here $w(.)$ and $||.||$ denote the numerical radius and the usual operator norm, respectively.

Keywords: Numerical radius, Usual operator norm, Operator matrix, Buzano ′s inequality.

Introduction

Let $\mathcal{B}(\mathcal{H})$ be the C^* −algebra of all bounded linear operators on the complex Hilbert space $\mathcal{H}$. For $T\in\mathcal{B}(\mathcal{H})$, the numerical radius $w(.)$ and the usual operator norm $||.||$ are, respectively, defined by

$$w(A)=\sup_{||x||=1}|\langle Ax,x\rangle| \text{ and } \|A\|=\sup_{||x||=1}\|Ax\|.$$

It is clear that $w(.)$ defines a norm on $\mathcal{B}(\mathcal{H})$. Moreover, it is known that $w(.)$ is equivalent to the usual operator norm $||.||$ on $\mathcal{B}(\mathcal{H})$ and with the following two sided inequality

$$\frac{1}{2}\|A\|\le w(A)\le\|A\| \text{ for every } A\in\mathcal{B}(\mathcal{H}). \tag{1.1}$$

An important property for the numerical radius is the power inequality, which says that

$$w(A^n)\le w^n(A) \text{ for every } n\in\mathbb{N} \text{ and } A\in\mathcal{B}(\mathcal{H}).$$

In Kittaneh (2005), the author provided refinements of the bounds in (1.1) by showing that

$$\frac{1}{4}\||A|^2+|A^*|^2\|\le w^2(A)\le\frac{1}{2}\||A|^2+|A^*|^2\| \text{ for every } A\in\mathcal{B}(\mathcal{H}). \tag{1.2}$$

In El-Haddad and Kittaneh (2007) the authors provided a generalization for the second inequality in (1.2) by showing that

$$w^{2r}(A)\le\frac{1}{2}\||A|^{2r}+|A^*|^{2r}\| \text{ for every } r\ge 1 \text{ and } A\in\mathcal{B}(\mathcal{H}). \tag{1.3}$$

In Dragomir (2009), the author presented an important upper bound for the numerical radii of products of two operators by showing that if $A, B \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$, then

$$w^r(B^*A) \leq \frac{1}{2}\||A|^{2r} + |B|^{2r}\|. \tag{1.4}$$

Recently in Al-dolat and Kittaneh (2023), the authors gave another improvement for the inequality in (1.3) by showing that if $A \in \mathcal{B}(\mathcal{H})$, then

$$w^{2r}(A) \leq \frac{1+\alpha}{4}\||A|^{2r} + |A^*|^{2r}\| + \frac{1-\alpha}{2} w^r(A^2) \tag{1.5}$$

for every $\alpha \in [0,1]$ and $r \geq 1$.

The direct sum of $2-$copies of $\mathcal{H}$ is denoted by $\mathcal{H}^{(2)} = \mathcal{H} \oplus \mathcal{H}$. Due to this decomposition, any $T \in \mathcal{B}(\mathcal{H}^{(2)})$ can be represented as a 2×2 operator matrix of the form $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A, B, C, D \in \mathcal{B}(\mathcal{H})$. Moreover, if $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H}^{(2)}$, then Tx is defined by $Tx = T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{pmatrix}$. To learn more about the numerical radii of operator of matrices and their application in finding estimates for the zeros of complex polynomials, one cane refer to Abo-Omar and Kittaneh (2015), Al-Dolat et al. (2016), Bani-Domi and Kittaneh (2008), Bani-Domi and Kittaneh (2009), Bani Domi and Kittaneh (2012) and Hirzallah et al. (2011).

The goal of this paper is to present several new upper bounds for the numerical radii of 2×2 operator matrices, then to refine the inequalities in (1.3) based on those bounds. Moreover, we provide refinements of earlier numerical radius inequalities due to Al-Dolat and Kittaneh (2023).

Main Results

To achieve our goal, we recall some well-known lemmas in order to establish our results. The first lemma is a consequence of the spectral theorem and Jensen ′s inequality see Kittaneh (2015).

Lemma 2.1 *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator and $x \in \mathcal{H}$ be any unit vector. Then, for $r \geq 1$, we have*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle.$$

The second lemma deals with non-negative convex functions and positive operators, and it can be found in Aujla and Sivla (2003).

Lemma 2.2 *Let f be a non-negative convex function on $[0,\infty)$ and $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. Then*

$$\left\|f\left(\frac{A+B}{2}\right)\right\| \leq \left\|\frac{f(A)+f(B)}{2}\right\|.$$

In particular,

$$\|(A+B)^r\| \leq 2^{r-1}\|A^r + B^r\| \quad \text{for every } r \geq 1.$$

The third lemma relates to certain 2×2 operator matrices and can be found in Hirzallah, Kittaneh and Shebrawi (2011).

Lemma 2.3 *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

(a) $w\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \max\{w(A), w(B)\};$

(b) $w\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = \max\{w(A+B), w(A-B)\}$.

In particular,

$$w\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) = w(B).$$

The next two lemmas can be found in Al-Dolat and Kittaneh (2023).

Lemma 2.4 *Let* $x, y, z \in \mathcal{H}$ *with* $\|z\| = 1$. *Then*

$$|\langle x,z\rangle\langle z,y\rangle|^r \leq \frac{1+\alpha}{2}\|x\|^r\|y\|^r + \frac{1-\alpha}{2}|\langle x,y\rangle|^r$$

for every $\alpha \in [0,1]$ and $r \geq 1$.

Lemma 2.5 *Let* $x, y, z \in \mathcal{H}$ *with* $||z|| = 1$. *Then*

$$|\langle x,z\rangle\langle z,y\rangle|^2 \leq \frac{1+\alpha}{4}\|x\|^2\|y\|^2 + \frac{1-\alpha}{4}|\langle x,y\rangle|^2 + \frac{1}{2}\|x\|\|y\||\langle x,y\rangle|$$

for every $\alpha \in [0,1]$.

The final lemma can be found in Al-Dolat and Al-Zoubi.

Lemma 2.6 *Let* $S, R \in \mathcal{B}(\mathcal{H})$ *be positive operators and let* $q \geq 1$. *Then*

$$\sup_{x\in\mathcal{H},||x||=1}(\langle Sx,x\rangle^q\langle Rx,x\rangle^q) \leq \frac{1}{4}\|S^{2q}+R^{2q}\| + \frac{1}{2}\min\{w(S^qR^q), w(S^qR^q)\}.$$

We begin our results by the following theorem, which provides a new upper bound for the numerical radii of 2×2 operator matrices, that will be used to give a refinement of the inequality (1.3).

Theorem 2.7 *Let* $B, C \in \mathcal{B}(\mathcal{H})$. *Then*

$$w^{2r}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1+\alpha}{8}\max\{\||C|^{2r}+|B^*|^{2r}\|, \||B|^{2r}+|C^*|^{2r}\|\}$$

$$+\frac{1+\alpha}{4}\min\{\max\{w(|C|^r|B^*|^r), w(|B|^r|C^*|^r)\}, \max\{w(|B^*|^r|C|^r), w(|C^*|^r|B|^r)\}\}$$

$$+\frac{1-\alpha}{2}\max\{w^r(BC), w^r(CB)\}$$

for every $\alpha \in [0,1]$ and $r \geq 1$.

Proof. Let $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ and let $x \in \mathcal{H}^{(2)}$ be any unit vector. Then we have

$$|\langle Tx,x\rangle|^{2r} = |\langle Tx,x\rangle\langle x,T^*x\rangle|^r$$

$$\leq \frac{1+\alpha}{2}\|Tx\|^r\|T^*x\|^r + \frac{1-\alpha}{2}|\langle Tx,T^*x\rangle|^r \quad \text{(by Lemma 2.4)}$$

$$= \frac{1+\alpha}{2}\langle |T|^2x,x\rangle^{\frac{r}{2}}\langle |T^*|^2x,x\rangle^{\frac{r}{2}} + \frac{1-\alpha}{2}|\langle T^2x,x\rangle|^r.$$

Thus,

$$w^{2r}(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|^{2r}$$

$$\leq \frac{1+\alpha}{2} \sup_{\|x\|=1} \left(\langle |T|^2 x, x\rangle^{\frac{r}{2}} \langle |T^*|^2 x, x\rangle^{\frac{r}{2}}\right) + \frac{1-\alpha}{2} w^r(T^2)$$

$$\leq \frac{1+\alpha}{8} \||T|^{2r} + |T^*|^{2r}\| + \frac{1+\alpha}{4} \min\{w(|T|^r|T^*|^r), w(|T^*|^r|T|^r)\} + \frac{1-\alpha}{2} w^r(T^2)$$

$$= \frac{1+\alpha}{8} \left\|\begin{bmatrix} |C|^{2r} + |B^*|^{2r} & 0 \\ 0 & |B|^{2r} + |C^*|^{2r} \end{bmatrix}\right\|$$

$$+ \frac{1+\alpha}{4} \min\left\{ w\left(\begin{bmatrix} |C|^r|B^*|^r & 0 \\ 0 & |B|^r|C^*|^r \end{bmatrix}\right), w\left(\begin{bmatrix} |B^*|^r|C^*|^r & 0 \\ 0 & |C^*|^r|B^*|^r \end{bmatrix}\right) \right\}$$

$$+ \frac{1-\alpha}{2} w^r\left(\begin{bmatrix} BC & 0 \\ 0 & CB \end{bmatrix}\right)$$

$$\leq \frac{1+\alpha}{8} \max\{\||C|^{2r} + |B^*|^{2r}\|, \||B|^{2r} + |C^*|^{2r}\|\}$$

$$+ \frac{1+\alpha}{4} \min\{\max\{w(|C|^r|B^*|^r), w(|B|^r|C^*|^r)\}, \max\{w(|B^*|^r|C|^r), w(|C^*|^r|B|^r)\}\}$$

$$+ \frac{1-\alpha}{2} \max\{w^r(BC), w^r(CB)\}$$

This completes the proof of the theorem.

As a direct consequence of the above theorem we have the following refinement of the inequality (1.5).

Corollary 2.8 *Let* $A \in \mathcal{B}(\mathcal{H})$*. Then*

$$w^{2r}(A) \leq \frac{1+\alpha}{8} \||A|^{2r} + |A^*|^{2r}\| + \frac{1+\alpha}{4} w(|A|^r|A^*|^r) + \frac{1-\alpha}{2} w^r(A^2)$$

$$\leq \frac{1+\alpha}{4} \||A|^{2r} + |A^*|^{2r}\| + \frac{1-\alpha}{2} w^r(A^2)$$

for every $\alpha \in [0,1]$ and $r \geq 1$.

Proof. By letting $B = C = A$ in Theorem 2.7, we get

$$w^{2r}(A) \leq w^{2r}\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) \quad \text{(by Lemma2.1)}$$

$$\leq \frac{1+\alpha}{8} \||A|^{2r} + |A^*|^{2r}\| + \frac{1+\alpha}{4} w(|A|^r|A^*|^r) + \frac{1-\alpha}{2} w^r(A^2) \quad \text{(by Theorem 2.7)}$$

$$\leq \frac{1+\alpha}{8} \||A|^{2r} + |A^*|^{2r}\| + \frac{1+\alpha}{8} \||A|^{2r} + |A^*|^{2r}\| + \frac{1-\alpha}{2} w^r(A^2) \quad \text{(by the inequality (1.4))}$$

$$= \frac{1+\alpha}{4} \||A|^{2r} + |A^*|^{2r}\| + \frac{1-\alpha}{2} w^r(A^2)$$

Remark 2.9 *The upper bound in Corollary 2.8 is a refinement of [Al-Dolat and Kittaneh (2023), Theorem 2.7], namely*

$$w^{2r}\left(\begin{bmatrix}0 & B\\ C & 0\end{bmatrix}\right) \leq \frac{1+\alpha}{4}\max\{\||B|^{2r}+|C^*|^{2r}\|, \||C|^{2r}+|B^*|^{2r}\|\} + \frac{1-\alpha}{2}\max\{w^r(CB), w^r(BC)\}$$

To explain this, note that

$$w^{2r}\left(\begin{bmatrix}0 & B\\ C & 0\end{bmatrix}\right) \leq \frac{1+\alpha}{8}\max\{\||B|^{2r}+|C^*|^{2r}\|, \||C|^{2r}+|B^*|^{2r}\|\}$$

$$+\frac{1+\alpha}{4}\min\left\{\max\{w(|C|^r|B^*|^r), w(|B|^r|C^*|^r)\}, \max\{w(|B^*|^r|C|^r), w(|C^*|^r|B|^r)\}\right\}$$

$$+\frac{1-\alpha}{2}\max\{w^r(CB), w^r(BC)\}$$

$$\leq \frac{1+\alpha}{8}\max\{\||B|^{2r}+|C^*|^{2r}\|, \||C|^{2r}+|B^*|^{2r}\|\}$$

$$+\frac{1+\alpha}{8}\max\{\||B|^{2r}+|C^*|^{2r}\|, \||C|^{2r}+|B^*|^{2r}\|\}$$
$$+\frac{1-\alpha}{2}\max\{w^r(CB), w^r(BC)\} \quad \text{(by the inequality (1.4))}$$

$$= \frac{1+\alpha}{4}\max\{\||B|^{2r}+|C^*|^{2r}\|, \||C|^{2r}+|B^*|^{2r}\|\} + \frac{1-\alpha}{2}\max\{w^r(CB), w^r(BC)\}.$$

We are now in a position to prove our next result.

Theorem 2.10 *Let* $B, C \in \mathcal{B}(\mathcal{H})$. *Then*

$$w^4\left(\begin{bmatrix}0 & B\\ C & 0\end{bmatrix}\right) \leq \frac{1+\alpha}{16}\max\{\||B|^4+|C^*|^4\|, \||C|^4+|B^*|^4\|\}$$

$$+\frac{1+\alpha}{8}\min\left\{\max\{w(|C|^2|B^*|^2), w(|B|^2|C^*|^2)\}, \max\{w(|B^*|^2|C|^2), w(|C^*|^2|B|^2)\}\right\}$$

$$+\frac{1-\alpha}{4}\max\{w^2(CB), w^2(BC)\}$$

$$+\frac{1}{4}\max\{w(CB), w(BC)\}\max\{\||B|^2+|C^*|^2\|, \||C|^2+|B^*|^2\|\}$$

for every $\alpha \in [0,1]$.

Proof. Let $T = \begin{bmatrix}0 & B\\ C & 0\end{bmatrix}$ and let $x \in \mathcal{H}^{(2)}$ be any vector. Then

$$|\langle Tx, x\rangle|^4 = |\langle Tx, x\rangle\langle x, T^*x\rangle|^2$$

$$\leq \frac{1+\alpha}{4}\|Tx\|^2\|T^*x\|^2 + \frac{1-\alpha}{4}|\langle Tx, T^*x\rangle|^2 + \frac{1}{2}\|Tx\|\|T^*x\||\langle Tx, T^*x\rangle|$$

$$\leq \frac{1+\alpha}{4}(|\langle |T|^2x, x\rangle||\langle |T^*|^2x, x\rangle|) + \frac{1-\alpha}{4}|\langle T^2x, x\rangle|^2 + \frac{1}{4}|\langle T^2x, x\rangle|(\|Tx\|^2 + \|T^*x\|^2)$$
(by the triangle inequality)

$$= \frac{1+\alpha}{4}(|\langle |T|^2x, x\rangle||\langle |T^*|^2x, x\rangle|) + \frac{1-\alpha}{4}|\langle T^2x, x\rangle|^2 + \frac{1}{4}|\langle T^2x, x\rangle|\langle(|T|^2 + |T^*|^2)x, x\rangle.$$

Thus,

$$w^4(T) = \sup_{\|x\|=1}|\langle Tx, x\rangle|^4$$

$$\leq \frac{1+\alpha}{4}\sup_{\|x\|=1}(|\langle|T|^2x,x\rangle||\langle|T^*|^2x,x\rangle|)+\frac{1-\alpha}{4}w^2(T^2)+\frac{1}{4}w(T^2)\||T|^2+|T^*|^2\|$$

$$\leq \frac{1+\alpha}{16}\||T|^4+|T^*|^4\|+\frac{1+\alpha}{8}\min\{w(|T|^2|T^*|^2),w(|T^*|^2|T|^2)\}$$

$$+\frac{1-\alpha}{4}w^2(T^2)+\frac{1}{4}w(T^2)\||T|^2+|T^*|^2\|$$

$$=\frac{1+\alpha}{16}\max\{\||B|^4+|C^*|^4\|,\||C|^4+|B^*|^4\|\}$$

$$+\frac{1+\alpha}{8}\min\left\{\max\{w(|C|^2|B^*|^2),w(|B|^2|C^*|^2)\},\max\{w(|B^*|^2|C|^2),w(|C^*|^2|B|^2)\}\right\}$$

$$+\frac{1-\alpha}{4}\max\{w^2(CB),w^2(BC)\}$$

$$+\frac{1}{4}\max\{w(CB),w(BC)\}\max\{\||B|^2+|C^*|^2\|,\||C|^2+|B^*|^2\|\}.$$

As a special case of Theorem 2.10 we have the following refinement of the inequality (1.3) for the special $r=2$.

Corollary 2.11 *Let* $A\in\mathcal{B}(\mathcal{H})$ *and let* $\alpha\in[0,1]$. *Then*

$$w^4(A)\leq\frac{1+\alpha}{16}\||A|^4+|A^*|^4\|+\frac{1+\alpha}{8}\min\{w(|A|^2|A^*|^2),w(|A^*|^2|A|^2)\}$$

$$+\frac{1-\alpha}{4}w^2(A^2)+\frac{1}{4}w(A^2)\||A|^2+|A^*|^2\|$$

$$\leq\frac{1}{2}\||A|^4+|A^*|^4\|.$$

Proof. Let $B=C=A$ in the above theorem. Then we have

$$w^4(A)=w\left(\begin{bmatrix}0 & A\\ A & 0\end{bmatrix}\right)\ \text{(by Lemma2.1)}$$

$$\leq\frac{1+\alpha}{16}\||A|^4+|A^*|^4\|+\frac{1+\alpha}{8}\min\{w(|A|^2|A^*|^2),w(|A^*|^2|A|^2)\}$$

$$+\frac{1-\alpha}{4}w^2(A^2)+\frac{1}{4}w(A^2)\||A|^2+|A^*|^2\|\ \text{(by Theorem 2.10)}$$

$$\leq\frac{1+\alpha}{16}\||A|^4+|A^*|^4\|+\frac{1+\alpha}{16}\||A|^4+|A^*|^4\|+\frac{1-\alpha}{8}\||A|^4+|A^*|^4\|\quad+\frac{1}{8}\||A|^2+|A^*|^2\|^2$$
(by the inequality (1.4))

$$=\frac{1}{4}\||A|^4+|A^*|^4\|+\frac{1}{8}\|(|A|^2+|A^*|^2)^2\|$$

(by the fact: If $X\in\mathcal{B}(\mathcal{H})$ is normaland $n\in\mathbb{N}$, then $||X^n||=||X||^n$)

$$\leq\frac{1}{4}\||A|^4+|A^*|^4\|+\frac{1}{4}\||A|^4+|A^*|^4\|\ \text{(by Lemma 2.2)}$$

$$=\frac{1}{2}\||A|^4+|A^*|^4\|.$$

At the end of this paper, we remark that the upper bound obtained in Theorem 2.10 is better than the upper bound given in Al-Dolat and Kittaneh (2023), (Theorem 2.12).

Scientific Ethics Declaration

The author declares that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the author.

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