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On Graded 2-n-Submodules of Graded Modules Over Graded Commutative Rings

Khaldoun Al-Zoubi
Jordan University of Science and Technology

Abstract: In this article, all rings are commutative with a nonzero identity. Let G be a group with identity e, R be a G-graded commutative ring, and M be a graded R-module. The concept of graded n-ideals was introduced and studied by Al-Zoubi et al. (2019). A proper graded ideal I of R is said to be a graded n-ideal of R if whenever $r, s \in h(R)$ with $rs \in I$ and $r \notin Gr(0)$, then $s \in I$. Recently the notion of graded n-ideals was extended to graded n-submodules by Al-Azaizeh and Al-Zoubi (2023). A proper graded submodule N of a graded R-module M is said to be a graded n-submodule if whenever $t \in h(R)$, $m \in h(R)$ with $tm \in N$ and $t \notin Gr(Ann_R(M))$, then $m \in N$. In this study, we introduce the concept of graded 2-n-submodules of graded modules over graded commutative rings, generalizing the concept of graded n-submodules. We investigate some characterizations of graded 2-n-submodules and investigate the behavior of this structure under graded homomorphism and graded localization. A proper graded submodule U of M is said to be a graded 2-n-submodule if whenever $r, s \in h(R)$, $m \in (M)$ and $rsm \in U$, then $rs \in Gr(Ann_R(M))$ or $rm \in U$ or $tm \in U$..

Keywords: Graded 2-n-submodules, Graded n-submodule, Graded 2-n ideals, Graded 2-nil-ideals

Introduction

Throughout this article, we assume that R is a commutative G-graded ring with identity and M is a unitary graded R-module. The concept of graded primary ideal was introduced and studied in Refai and Al-Zoubi (2004). In Al-Zoubi et al. (2019), the concept of graded 2-absorbing ideals was introduced and studied as a generalization of graded prime ideals. The concept of graded 2-absorbing submodules was introduced and studied in Al-Zoubi and Abu-Dawwas (2014). In Al-Zoubi et al. (2017), the concept of graded 2-absorbing primary ideals was presented and studied. As a generalization of graded 2-absorbing primary ideals, the authors introduced and studied graded 2-absorbing primary submodules in Celikel (2016). In 2019, different type of graded ideal, namely, graded n-ideal, was introduced in Al-Zoubi and Al-Turman (2019). Recently, in Al-Azaizeh and Al-Zoubi (2023), the notion of graded n-ideals was extended to graded n-submodules. In this paper, we introduce the concept of graded 2-n-submodules of graded modules over graded commutative rings, generalizing the concept of graded n-submodules. A number of results concerning graded 2-n-submodules are given. As an example, we characterized graded 2-n-submodules and studied graded 2-n-submodules under graded homomorphism and under graded localization.

Preliminaries

In this section we will give the definitions and results which are required in the next section.

Definition 2.1.

1. Let G be a group with identity e and R be a commutative ring with identity 1_R. Then R is G-graded ring if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g. The set of all homogeneous elements of R is denoted by $h(R)$, i.e. $h(R) = \bigcup_{g \in G} R_g$, see (Nastasescu et al., 2004).

2. Let $R = \bigoplus_{g \in G} R_g$ be G-graded ring, an ideal P of R is called a graded ideal if $P = \sum_{h \in G} P \cap R_h = \sum_{h \in G} P_h$, see (Nastasescu et al., 2004).

3. A left R-module M is said to be a G-graded R-module if $M = \bigoplus_{g \in G} M_g$ with $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$, where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. The set of all homogeneous elements of M is denoted by $h(M)$, i.e, $h(M) = \bigcup_{g \in G} M_g$. Note that M_h is an R_e-module for every $h \in G$. For more properties, see (Nastasescu et al., 2004).

4. A submodule K of M is called a graded submodule of M if $K = \bigoplus_{h \in G} (K \cap M_h) := \bigoplus_{h \in G} K_h$, see (Nastasescu et al., 2004).

5. If K is graded submodule of M, then $(K :_R M) = \{a \in R \mid aM \subseteq K\}$ is graded ideal of R. Furthermore, the annihilator of K in R is denoted and defined by $Ann_R(K) = \{a \in R \mid aK = \{0\}\}$, see (Atani, 2006).

Definition 2.2.

1. A proper graded submodule U of M is said to be a graded n-submodule (briefly, gr-n-submodule) if whenever $r \in h(R)$, $m \in h(M)$ with $rm \in U$ and $r \notin Gr(Ann_R(M))$, then $m \in U$, see (Al-Azaizeh & Al-Zoubi, 2023).

2. The graded radical of a graded ideal I, denoted by $Gr(I)$, is the set of all $t = \sum_{g \in G} t_g \in R$ such that for each $g \in G$ there exists $n_g \in \mathbb{N}$ with $t_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$, see (Refai & Al-Zoubi, 2004).

3. A proper graded submodule P of M is called a graded prime (briefly, gr-prime) submodule if whenever $a \in h(R)$ and $m \in h(M)$ with $am \in P$, then either $a \in (P :_R M)$ or $m \in P$, see (Atani, 2006).

4. A proper graded submodule U of M is called graded primary (briefly, gr-primary) submodule if $rm \in U$, then either $m \in U$ or $r \in Gr((U :_R M))$, where $r \in h(R)$ and $m \in h(M)$, see (Oral et al., 2011)

5. A proper graded submodule N of M is said to be a graded 2-absorbing (briefly, gr-2-absorbing) submodule of M if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rm \in N$ or $sm \in N$ or $rs \in (N :_R M)$, see (Al-Zoubi & Abu-Dawwas, 20014).

6. A proper graded submodule U of M is called graded primary (briefly, gr-primary) submodule if $rm \in U$, then either $m \in U$ or $r \in Gr((U :_R M))$, where $r \in h(R)$ and $m \in h(M)$, see (Oral et al., 2011).

7. The graded radical of a graded submodule N of M, denoted by $Gr_M(N)$, is defined to be the intersection of all graded prime submodules of M containing N. If N is not contained in any graded prime submodule of M, then $Gr_M(N) = M$, see (Atani &Farzalipour, 2007).

8. A proper graded submodule N of M is said to be a graded 2-absorbing (briefly, gr-2-absorbing) primary submodule of M if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then $rs \in (N :_R M)$ or $rm \in Gr_M(N)$ or $sm \in Gr_M(N)$, see (Celikel, 2016).

9. A proper graded ideal I of R is said to be a graded n-ideal (briefly, gr-n-ideal) of R if whenever $r, s \in h(R)$ with $rs \in I$ and $r \notin Gr(0)$, then $s \in I$, see (Al-Zoubi & Al-Turman, 2019).

10. A proper graded ideal P of R is said to be graded 2-nil (briefly, gr-2-nil) ideal if whenever, $a, b, c \in h(\Im)$ with $abc \in P$, then $ab \in Gr(0)$ or $ac \in P$ or $bc \in P$, see (Abu Qayass &Al-Zoubi, 2023).

11. A graded R-module M is called a graded faithful (briefly, gr- faithful) module if $rM = 0$, then $r = 0$ for $r \in h(R)$, see (Atani, 2006).

12. Let R be a G-graded ring and M, M' be graded R-modules. Let $\varphi: M \to M'$ be an R-module homomorphism. Then φ is said to be a graded homomorphism if $\varphi(M_g) \subseteq M'_g$ for all $g \in G$, see (Nastasescu et al., 2004).

13. A nonzero graded module M is called a graded second (briefly, gr-second) module if $Ann_R(M) = Ann_R(M/N)$ for all graded submodule N of M, see (Ceken & Alkan, 2015)

Characterization of gr-2-n-submodule

In this section, we provide several characterizations of gr-2-n-submodule. We begin by introducing the notion of gr-2-n-submodules.

Definition 3.1 *A proper graded submodule U of M is said to be a graded 2-n-submodule (briefly, gr-2-n-submodule) of M if whenever $r, s \in h(R)$, $m \in (M)$ and $rsm \in U$, then $rs \in Gr(Ann_R(M))$ or $rm \in U$ or $tm \in U$.*

From this previous definition, we can conclude that: every gr-n-submodule is a gr-2-n-submodule, every gr-prime submodule is gr-2-n-submodule. gr-2-n-submodule is a gr-2-absorbing primary submodule. Also, every gr-2-n-submodule U of a graded R-module M satisfying $Gr(Ann_R(M) \subseteq (U:_R M)$ is gr-2-absorbing. The following example shows that these notions are different in general.

Example 3.2 Let p and q be prime integers. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$. Then R is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}$. Then M is a graded R-module with $M_0 = \mathbb{Z}$ and $M_1 = \{0\}$.

(a) Consider the graded submodule $U = p\mathbb{Z}$ of M. Then U is a gr-2-n-submodule since it is gr-prime. Since $p.1 \in U$ but $1 \notin U$ and $p \notin Gr(Ann_R(\mathbb{Z})) = \{0\}$, U is not gr-n-submodule.

(b) Consider the graded submodule $U = pq\mathbb{Z}$ of M. Then U is a gr-2-absorbing (gr-2-absorbing primary) submodule of M. Since $p.q.1 \in U$, but $p.1 \notin U$, $q.1 \notin U$, and $p.q \notin Gr(Ann_R(\mathbb{Z})) = \{0\}$, U is not a gr-2-n-submodule of M.

(c) Let $G = \mathbb{Z}_2$ and $\Re = \mathbb{Z}$. Then $\Re$ is a G-graded ring with $\Re_0 = \mathbb{Z}$ and $\Re_1 = \{0\}$. Let $\Im = \mathbb{Z}_{p^3}$. Then $\Im$ is a graded $\Re$-module with $\Im_0 = \mathbb{Z}_{p^3}$ and $\Im_1 = \{0\}$. Consider the graded submodule $U = p^2\mathbb{Z}_{p^3}$ of $\mathbb{Z}$-module $\mathbb{Z}_{p^3}$. Then U is a gr-2-n-submodule of $\mathbb{Z}_{p^3}$ that is not gr-prime.

From the definitions above, the following theorem follows immediately.

Theorem 3.3 *Let U be a proper graded submodule of graded R-module*

1. If U is a gr-primary submodule and $Gr(Ann_R(M)) = (U:_R M)$, then U is a gr-2-n-submodule of M.
2. If M is a gr-second module, then the two concepts of gr-2-n-submodule and gr-2-absorbing primary submodules coincide.
3. If $Gr(Ann_R(M)) = (U:_R M)$, then the following are equivalent : [(a)]

(a) U is a gr-2-n-submodule of M.
(b) U is a gr-2-absorbing submodule of M.
(c) U is a gr-2-absorbing primary submodule of M

As shown in the following example, the condition $Gr(Ann_R(M)) = (U:_R M)$ in Theorem 3.3(1) is crucial:

Example 3.4 *Let p be prime integer. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$. Then R is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}$. Then M is a graded R-module with $M_0 = \mathbb{Z}$ and $M_1 = \{0\}$. Consider the graded submodule $U = p^2\mathbb{Z}$ of M. Then U is a gr-primary, but it is not gr-2-n-submodule of M as $p.p.1 \in U$ but neither $p.p \in Gr(Ann_R(\mathbb{Z})) = \{0\}$ nor $p.1 \in U$.*

Let U be a graded submodule of M and $r \in h(R)$. We use the notation $(U:_M r)$ to denote the graded submodule $\{m \in M: rm \in U\}$ of M.

Theorem 3.5 *Let U be a proper graded submodule of M. Then the following statements are equivalent:*

(i) U is a gr-2-n-submodule of M.
(ii) If $r, s \in h(R)$ and $rs \notin Gr(Ann_R(M))$, then $(U:_M rs) \subseteq (U:_M r) \cup (U:_M s)$.
(iii) If $r, s \in h(R)$ and $rs \notin Gr(Ann_R(M))$, then $(U:_M rs) \subseteq (U:_M r)$ or $(U:_M rs) \subseteq (U:_M s)$.

Proof.

$(i) \Rightarrow (ii)$ Let U be a gr-2-n-submodule of M, $r, s \in h(R)$ such that $rs \notin Gr(Ann_R(M))$. Let $m \in (U:_M rs) \cap h(M)$. Then $rsm \in U$. Since U is a gr-2-n-submodule of M, $rsm \in U$ and $rs \notin Gr(Ann_R(M))$, we have either $rm \in U$ or $sm \in U$. Thus $m \in (U:_M r)$ or $m \in (U:s)$. This follows $(U:_M rs) \subseteq (U:_M r) \cup (U:_M s)$.

$(ii) \Rightarrow (iii)$ If a graded submodule is a subset of the union of two graded submodules, then it is a subset of one of them by (Atani & Tekir, 2007, Lemma 2.2). Thus we get the result.

$(iii) \Rightarrow (i)$ Let $r, s \in h(R)$ and $m \in h(M)$ such that $rsm \in U$ and $rs \notin Gr(Ann_R(M))$. Then $m \in (U:_M rs) \subseteq (U:_M r)$ or $m \in (U:_M rs) \subseteq (U:_M s)$ by (iii), and so $rm \in U$ or $sm \in U$.

We give another characterisation of gr-2-n-submodules in the following theorem.

Theorem 3.6 *Let U be a gr-2-n-submodule of M. Let $V = \bigoplus_{g\in G} V_g$ be a graded submodule of M, $I = \bigoplus_{h\in G} I_h$ and $J = \bigoplus_{l\in G} J_l$ be two graded ideals of M. Then the following statements are equivalent:*

(i) U is a gr-2-n-submodule of M.
(ii) For every $g \in G$, if $rsV_g \subseteq U$ for some $r, s \in h(R)$, then either $rs \in Gr(Ann_R(M))$ or $rV_g \subseteq U$ or $sV_g \subseteq U$.
(iii) For every $g, h, l \in G$, if $I_h J_l V_g \subseteq U$, then either $I_h J_l \subseteq Gr(Ann_R(M))$ or $I_h V_g \subseteq U$ or $J_l V_g \subseteq U$.

Proof.

$(i) \Rightarrow (ii)$ Let U be a gr-2-n-submodule of M, $r, s \in h(R)$ and $g \in G$ such that $rsV_g \subseteq U$. Assume on the contrary that $rs \notin Gr(Ann_R(M))$, $rV_g \nsubseteq U$ and $sV_g \nsubseteq U$. Then there exist $v_g, v'_g \in V_g$ with $rv_g \notin U$ and $sv'_g \notin U$. Since U is a gr-2-n-submodule of M, $rsv_g \in U$, $rv_g \notin U$ and $rs \notin Gr(Ann_R(M))$, we have $sv_g \in U$. Similarly, sinice $rsv'_g \in U$, $sv'_g \notin U$ and $rs \notin Gr(Ann_R(M))$, we have $rv'_g \in U$. By $(v_g + v'_g) \in V_g$, we get $rs(v_g + v'_g) \in U$. Then either $r(v_g + v'_g) \in U$ or $s(v_g + v'_g) \in U$ as U is a gr-2-n-submodule of M and $rs \notin Gr(Ann_R(M))$. If $r(v_g + v'_g) = rv_g + rv'_g \in U$, then $rv_g \in U$, a contradiction. If $s(v_g + v'_g) = sv_g + sv'_g \in U$, then $sv'_g \in U$, a contradiction. Thus we get the result.

$(ii) \Rightarrow (iii)$ Assume that (ii) holds. Let $g, h, l \in G$ such that $I_h J_l V_g \subseteq U$. Assume that $I_h J_l \nsubseteq Gr(Ann_R(M))$, $I_h V_g \nsubseteq U$ and $J_l V_g \nsubseteq U$. Then there exist $i_h, i'_h \in I_h$ and $j_l, j'_l \in J_l$ such that $i_h V_g \nsubseteq U$, $j_l V_g \nsubseteq U$ and $i'_h j'_l \notin Gr(Ann_R(M))$. Since $i_h j_l V_g \subseteq U$, $i_h V_g \nsubseteq U$ and $j_l V_g \nsubseteq U$, we have $i_h j_l \in Gr(Ann_R(M))$. Now since $i'_h j'_l V_g \subseteq U$ and $i'_h j'_l \notin Gr(Ann_R(M))$, we have either $i'_h V_g \subseteq U$ or $j'_l V_g \subseteq U$ by (ii). We consider three cases:

Case 1: Assume that $i'_h V_g \subseteq U$ but $j'_l V_g \nsubseteq U$. Since $i_h j'_l V_g \subseteq U$, $j'_l V_g \nsubseteq U$ and $i_h V_g \nsubseteq U$, we have $i_h j'_l \in Gr(Ann_R(M))$. Since $i'_h V_g \subseteq U$ but $i_h V_g \nsubseteq U$, we have $(i_h + i'_h) V_g \nsubseteq U$. By $(i_h + i'_h) j'_l V_g \subseteq U$, $(i_h + i'_h) V_g \nsubseteq U$ and $j'_l V_g \nsubseteq U$, we get $(i_h + i'_h) j'_l = i_h j'_l + i'_h j'_l \in Gr(Ann_R(M))$. Then $i'_h j'_l \in Gr(Ann_R(M))$, a contradiction.

Case 2: Assume that $j'_l V_g \subseteq U$ but $i'_h V_g \nsubseteq U$, similar to Case 1.

Case 3: Assume that $i'_h V_g \subseteq U$ and $j'_l V_g \subseteq U$. By $j'_l V_g \subseteq U$ and $j_l V_g \nsubseteq U$, we have $(j_l + j'_l) V_g \nsubseteq U$. Since $i_h (j_l + j'_l) V_g \subseteq U$, $(j_l + j'_l) V_g \nsubseteq U$ and $i_h V_g \nsubseteq U$, we have $i_h (j_l + j'_l) = i_h j_l + i_h j'_l \in Gr(Ann_R(M))$. Then $i_h j'_l \in Gr(Ann_R(M))$ since $i_h j_l \in Gr(Ann_R(M))$. In the same way as above, since $i'_h V_g \subseteq U$ and $i_h V_g \nsubseteq U$, we get $(i_h + i'_h) V_g \nsubseteq U$ and $i'_h j_l \in Gr(Ann_R(M))$.Now, since $(i_h + i'_h)(j_l + j'_l) V_g \subseteq U$, $(i_h + i'_h) V_g \nsubseteq U$ and $(j_l + j'_l) V_g \nsubseteq U$, we have$(i_h + i'_h)(j_l + j'_l) = i_h j_l + i_h j'_l + i'_h j_l + i'_h j'_l \in Gr(Ann_R(M))$ and hence $i'_h j'_l \in Gr(Ann_R(M))$, a contradiction.

$(iii) \Rightarrow (i)$ Let $r_h, s_l \in h(R)$ and $m_g \in h(M)$ with $r_h s_l m_g \in U$. Let $I = (r_h)$ and $J = (s_l)$ be a graded ideals of R generated by r_h, s_l, respectively. Let $V = R m_g$ be a graded submodule of M generated by m_g . Then $I_h J_l V_g \subseteq U$ and hence either $I_h J_l \subseteq Gr(Ann_R(M))$ or $I_h V_g \subseteq U$ or $J_l V_g \subseteq U$. Hence either $r_h m_g \in U$ or $s_l m_g \in U$ or $r_h s_l \in Gr(Ann_R(M))$. Thus U is a gr-2-n-submodule of M.

Theorem 3.7 *Let U be a proper graded submodule of M. Then*

(i) If $(U :_R m)$ is gr-n-ideal of R for all $m \in h(M) \backslash U$, then U is a gr-2-n-submodule of M.
(ii) If U is a gr-2-n-submodule of M, then $(U :_M a)$ is a gr-2-n-submodule of M containing U for all $a \in h(R) \backslash (U :_R M)$.
(iii) Let M be a gr-faithful R-module. If U is a gr-2-n-submodule of M, then $(U :_R m)$ is a gr-nil deal of R containing $(U :_R M)$ for all $m \in h(M) \backslash U$.

Proof.

(i) Let $rsm \in U$ and $rs \notin Gr(Ann_R(M))$ for some $r, s \in h(R)$ and $m \in h(M)$. Hence $rs \in (U :_R m)$ and $r, s \notin Gr(0)$. If $m \in U$, done. So assume that $m \notin U$. Then $r, s \in (U :_R m)$ as $(U :_R m)$ is gr-n-ideal of R. So $rm, sm \in U$, as required..

(ii) Let $a \in h(R) \backslash (U :_R M)$ and let $rsm \in (U :_M a)$ for some $r, s \in h(R)$ and $m \in h(M)$. Then $(U :_M a) \neq M$. Since U is a gr-2-n-submodule of M and $rsma \in U$, we have either $rs \in Gr(Ann_R(M))$ or $rma \in U$ or $sma \in U$. Then either $rs \in Gr(Ann_R(M))$ or $rm \in (U :_M a)$ or $sm \in (U :_M a)$, as required.

(iii) Let $m \in h(M) \backslash U$. Hence $(U :_R m) \neq R$. Now, assume that $r, s, t \in h(R)$ with $rst \in (U :_R m)$ and $rs \notin Gr(0)$. Then $rs \notin Gr(Ann_R(M))$ as M is gr-faithful. Since U is a gr-2-n-submodule of M, $rstm \in U$ and $rs \notin Gr(Ann_R(M))$, we have either $rtm \in U$ or $stm \in U$, i.e. $rt \in (U :_R m)$ or $st \in (U :_R m)$.Therefore, $(U :_R m)$ is a gr-nil deal of R. Clearly, $(U :_R M) \subseteq (U :_R m)$.

Properties of gr-2-n-Submodules

Theorem 4.1 *Let M and M' be graded R-modules and $\varphi : M \to M'$ be graded R-module homomorphism.*

(i).Assume that φ is a graded monomorphism. If $U'$is gr-2-n-submodule of M' such that $\varphi^{-1}(U') \neq M$, then $\varphi^{-1}(U')$ is a gr-2-n-submodule of M.
(ii) Assume that φ is a graded epimorphism. If U is a gr-2-n-submodule of M with $Ker(\varphi) \subseteq U$, then $\varphi(U)$ is a gr-2-n-submodule of M'.

Proof.

(i) Suppose that $rsm \in \varphi^{-1}(U')$ and $rs \notin Gr(Ann_R(M)$ for some $r, s \in h(R)$ and $m \in h(M)$. Then $\varphi(rsm) = rs\varphi(m) \in U'$. As φ is a graded monomorphism and $rs \notin Gr(Ann_R(M)$, we get $rs \notin Gr(Ann_R(M')$. Since U' is gr-2-n-submodule of M', we have either $r\varphi(m) \in U'$ or $s\varphi(m) \in U'$. This implies that either $rm \in \varphi^{-1}(U')$ or $sm \in \varphi^{-1}(U')$. Therefore $\varphi^{-1}(U')$ is a gr-2-n-submodule of M.

(ii) Suppose that U is a gr-2-n-submodule of M with $Ker(\varphi) \subseteq U$. Let $rsm' \in \varphi(U)$ for some $r, s \in h(R)$ and $m' \in h(M')$. As φ is a graded epimorphism, then there exists $m \in h(M)$ such that $m' = \varphi(m)$, hence $rs\varphi(m) = \varphi(rsm) \in \varphi(U)$. Since $Ker(\varphi) \subseteq U$, we have $rsm \in U$. Since U is a gr-2-n-submodule of M, we have either $rm \in U$ or $sm \in U$ or $rs \in Gr(Ann_R(M)$ and so either $rm' = \varphi(rm) \in \varphi(U)$ or $sm' = \varphi(sm) \in \varphi(U)$ or $rs \in Gr(Ann_R(M'))$. Therefore, $\varphi(U)$ is a gr-2-n-submodule of M'.

Corollary 4.2 *Let $U \subseteq V$ be two graded submodules of M. Then the followings hold;*

(i) If V is a gr-2-n-submodule of M, then V/U is a gr-2-n-submodule of a graded R-module M/U.
(ii) If V/U is a gr-2-n-submodule of a graded R-module M/U and $(U:_R M) \subseteq Gr(Ann_R(M))$, then V is a gr-2-n-submodule of M.
(iii) Let U be a graded submodule of M. If V is a gr-2-n-submodule of M such that $U \nsubseteq V$, then $V \cap U$ is a gr-2-n-submodule of U.

Proof.

(i) Assume that V is a gr-2-n-submodule of M. Let $f: M \to M/U$ be a graded epimorphism defined by $f(m) = m + U$. Then $Ker(f) = U \subseteq V$, so by Theorem 4.1 (ii), V/U is a gr-2-n-submodule of M/U.

(ii) Is clear.

(iii) Assume that V is a gr-2-n-submodule of M such that $U \nsubseteq V$. Consider the graded monomorphism $\varphi: U \to M$ defined by $\varphi(m) = m$ for all $m \in M$. Then $\varphi^{-1}(V) = V \cap U$, so by Theorem 4.1 (i), $V \cap U$ is a gr-2-n-submodule of U.

Let I be a proper graded ideal of a G-graded ring R and N be a graded submodule of a graded R-module M. The notations $G\text{-}Z_I(R)$ and $G\text{-}Z_N(M)$ denote the sets $\{r \in h(R): rs \in I$ for some $s \in h(R)\backslash I\}$ and $\{r \in h(R): rm \in N$ for some $m \in h(M)\backslash N\}$.

The following result studies the behavior of a gr-2-n-submodules under localization.

Theorem 4.3 *Let $S \subseteq h(R)$ be a multiplication closed subset of R and U is a proper graded submodule of M.*

(i) If U is a gr-2-n-submodule of M with $(U:_R M) \cap S = \emptyset$, then $S^{-1}U$ is a gr-2-n-submodule of $S^{-1}M$.
(ii) Assume that $Gr(Ann_{S^{-1}R}(S^{-1}M)) = S^{-1}Gr(Ann_R(M))$. If $S^{-1}U$ is a gr-2-n-submodule of $S^{-1}M$, and $S \cap G\text{-}Z_{Gr(Ann_R(M))}(R) = S \cap G\text{-}Z_U(M) = \emptyset$, then U is a gr-2-n-submodule of M.

Proof.

(i) Assume that U is a gr-2-n-submodule of M with $(U:_R M) \cap S = \emptyset$. Let $\frac{r}{s_1}\frac{k}{s_2}\frac{m}{s_3} \in S^{-1}U$, where $\frac{r}{s_1}, \frac{k}{s_2} \in h(S^{-1}U)$ and $\frac{m}{s3} \in h(S^{-1}M)$. Then $trkm \in U$ for some $t \in S$. Since U is a gr-2-n-submodule of M, we have either $rk \in Gr(Ann_R(M))$ or $trm \in U$ or $tkm \in U$, which implies that either $\frac{r}{s_1}\frac{k}{s_2} \in S^{-1}Gr(Ann_R(M)) \subseteq Gr(Ann_{S^{-1}R}(S^{-1}M))$ or $\frac{r}{s_1}\frac{m}{s_3} = \frac{trm}{ts_1s_3} \in S^{-1}U$ or $= \frac{k}{s_2}\frac{m}{s_3} = \frac{tkm}{ts_2s_3} \in S^{-1}U$. Therefore, $S^{-1}U$ is a gr-2-n-submodule of $S^{-1}M$.

$ii)$ Let $rkm \in U$ for some $r, k \in h(R)$ and $m \in h(M)$. Then $\frac{r}{1}\frac{k}{1}\frac{m}{1} \in S^{-1}U$. Since $S^{-1}U$ is a gr-2-n-submodule of $S^{-1}M$, we have either $\frac{r}{1}\frac{k}{1} \in Gr(Ann_{S^{-1}R}(S^{-1}M)) = S^{-1}Gr(Ann_R(M))$ or $\frac{r}{1}\frac{m}{1} \in S^{-1}U$ or $\frac{k}{1}\frac{m}{1} \in S^{-1}U$. So, either $trk \in Gr(Ann_R(M)$ for some $t \in S$ or $lrm \in U$ for some $l \in S$ or $vkm \in U$ for some $v \in S$. This implies that either $rk \in Gr(Ann_R(M)$ or $rm \in U$ or $km \in U$ as $S \cap G\text{-}Z_{Gr(Ann_R(M))}(R) = S \cap G\text{-}Z_U(M) = \emptyset$. Therefore, U is a gr-2-n-submodule of M.

Let R be a commutative ring and M be an R-module. Then the idealization $R(+)M = \{(r,m)\colon r \in R$ and $m \in M\}$ is the ring whose elements are those of $R \times M$ equipped with addition and multiplication defined by $(r,m) + (r',m') = (r + r', m + m')$ and $(r,m)(r',m') = (rr', rm' + r'm)$ respectively. Let G be an abelian group, $R = \bigoplus_{g\in G} R_g$ be a G-graded ring and $M = \bigoplus_{g\in G} M_g$ be a G-grded R-module. Then $R(+)M$ is a G-graded ring with $(R(+)M)_g = R_g(+)M_g$, see (Uregen et al., 2019, Proposition 3.1). If I is an ideal of R and U is a submodule of M with $IM \subseteq U$. Then $I(+)U$ is a graded ideal of $R(+)M$ if and only if I is a graded ideal of R and U is a graded submodule of M, see (Uregen et al., 2019, Proposition 3.3).

Theorem 4.4 *Let I be a graded ideal of R and U be a proper graded submodule M. If $I(+)U$ is gr-2-nil ideal of $R(+)M$, then I is a gr-2-nil ideal of R and U is a gr-2-n-submodule of M.*

Proof.

Suppose that $I(+)U$ is a gr-2-nil ideal of $R(+)M$. At first we want to show that I is a gr-2-nil ideal of R. Let $rst \in I$ but $rs \notin Gr(0)$ for some $r, s,t \in h(R)$. Hence $(r,0_M)(s,0_M)(t,0_M) = (rst,0_M) \in I(+)U$ but $(r,0_M)(s,0_M) = (rs,0_M) \notin Gr(0_{R(+)M})$. Then either $(s,0_M)(t,0_M) = (st,0_M) \in I(+)U$ or $(r,0_M)(t,0_M) = (rt,0_M) \in I(+)U$ as $I(+)U$ is gr-2–nil ideal of $R(+)M$. This implies that either $st \in I$ or $rt \in I$. Thus I is a gr-nil ideal of R. Now, we want to show that U is a gr-2-n-submodule of M. Let $rsm \in U$ and $rs \notin Gr(Ann_R(M)$ for some $r, s \in h(R)$ and $m \in h(M)$. Then $(r,0_M)(s,0_M)(0,m) = (0,rsm) \in I(+)U$ with $(r,0_M)(s,0_M) \notin Gr(0_{R(+)M})$. Since $I(+)U$ is gr-2–nil ideal of $R(+)M$, we have either $(r,0_M)(0,m) = (0,rm) \in I(+)U$ or $(s,0_M)(0,m) = (0,sm) \in I(+)U$. Hence, either $rm \in U$ or $sm \in U$. Therefore, U is a gr-2-n-submodule of M.

Scientific Ethics Declaration

The author declares that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the author.

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Author Information

Khaldoun Al-Zoubi
Department of Mathematics and Statistics
Jordan University of Science and Technology
P.O.Box 3030, Irbid 22110, Jordan
Contact e-mail: *kfzoubi@just.edu.jo*

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