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Laplace Method for Calculate the Determinant of Cubic-Matrix of Order 2 and Order 3

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Abstract: In this paper, as a continuation of our work on the determinants of cubic matrices of order 2 and order 3, we have investigated the possibilities of developing the concept of determinants of cubic matrices with three indexes, as well as the possibility of calculating them using the Laplace expansion method. We have observed that the notion of permutation expansion, which is used for square determinants, and the concept of the Laplace expansion method, which is used for square and non-square (rectangular) determinants, may be applied to this novel concept of 3D determinants. In this research, we demonstrated that the Laplace expansion approach is also applicable to cubic matrices of the second and third orders. These results are presented simply and with extensive proof. The findings are also supported by illustrated cases. In addition, we provided an algorithmic explanation for the Laplace expansion approach applied to cubic matrices.

Keywords: Cubic-matrix determinant, Laplace expansion method, Permutation method, Computer algorithm.

Introduction

Linear Algebra, Abstract Algebra and Geometry are very intertwined fields. The applications of these fields are very important and useful. Considering that matrix theory has very important applications in *Computer Graphics*, *Game-Theory*, *Graph-Theory*, *Imagery*, *different problems from informatics*, *Partial differential equations*, etc., which are very important in many vital fields! We are trying to develop this further, introducing the concept of a 3-dimensional matrix and step by step to first study the determinants for a 3-dimensional matrix. More specifically, in this paper, we study the determinants of the cubic-matrix for order 2 and order 3. Based on the determinant of 2D square matrices presented in (Artin, 1991; Bretscher, 2018; Schneide et al., 1973; Lang, 2010), as well as the determinant of rectangular matrices presented in (Salihu & Marevci, 2021, Amiri et al., 2010; Radić, 1966; Radić, 2005; Makarewicz et al., 2014) we have come to the idea of developing the concept of the determinant of 3D cubic matrices in (Salihu-Zaka, 2023b), also in papers (Salihu-Zaka, 2023b) and (Salihu-Zaka, 2023a) we have studied and proved some basic properties related to the determinant of cubic-matrix of order 2 and 3. Also during this paper, we consider the results obtained in the papers (Amiri et al., 2010; Radić, 1966; Radić, 2005; Makarewicz et al., 2014; Salihu et al., 2022; Milne-Thomson, 1941; Gago et al., 2022; Kuloğlu et al., 2023), but also the results presented in books (Artin, 1991; Bretscher, 2018; Schneide et al., 1973; Lang, 2010; Poole, 2006; Rose, 2002). The history of Determinants and Linear Algebra, in general, is quite beautiful, for this we invite you to look (Eves, 1990; Grattan-Guinness, 2003) and some classic and very rich texts with knowledge about determinants, such as (Lang, 2002; Leon et al., 2021; Lay et al., 2022; Meyer, 2023; Muir, 1933; Price, 1947).

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In this paper, we study the properties of the determinants of the cubic-matrix of orders 2 and 3, related to the Laplace expansion method, our concept is based on the permutation expansion method. Encouraged by geometric intuition, in this paper, we are trying to give an idea and visualize the meaning of the determinants for the cubic-matrix. Our early research mainly lies between geometry, algebra, matrix theory, etc., (see (Peters-Zaka, 2023; Zaka & Peters, 2024; Zaka, 2019a; Zaka-Filipi, 2016; Filipi et al., 2019; Zaka, 2017; Zaka, 2018; Zaka, 2016; Zaka & Peters, 2019a; Zaka & Peters, 2019b; Zaka & Mohammed, 2020a; Zaka & Mohammed, 2020b). This paper is a continuation of the ideas that arise based on previous research of 3D matrix rings with elements from any whatever field (Zaka, 2017), but here we study the case when the field F is the field of real numbers $\mathbb{R}$ also is a continuation of our research (Salihu & Zaka, 2023b) related to the study of the properties of determinants for cubic-matrix of order 2 and 3. In this paper, we follow a different method from the calculation of determinants of the 3D matrix, which is studied in (Zaka, 2019b). In contrast to the meaning of the determinant as a multi-scalar studied in (Zaka, 2019b), in this paper, we give a new definition, for the determinant of the 3D-cubic-matrix, which is a real-number.

In the papers (Zaka, 2017; Zaka 2019b), have been studied in detail, properties for 3D-matrix, therefore, those studied properties are also valid for 3D-cubic-Matrix. Our point in this paper is to provide a concept of the determinant of 3D matrices using the Laplace concept which is a well-known methodology for calculating the determinant of square and rectangular matrices. Hence, our concept is based on the Laplace method which is used for calculating 2D square and rectangular determinants (Poole, 2006; Rose 2002) also, during this work we take into account the results achieved earlier, see (Rezaifar et al.i, 2007; Koprowski, 2022; Neto, 2015; Dutta & Pal, 2011; Silvester, 2000; Muir, 1906; Sothanaphan, 2018). At the end of this paper, we have also presented an algorithmic presentation for the Laplace expansion method, for the calculation of cubic-matrix-determinants.

Preliminaries

3D Matrix

The following is the definition of 3D matrices provided in 2017 in (Zaka 2017): see Fig.1 for 3-D matrix appearance

Definition 1 *3-dimensional $m \times n \times p$ matrix will call, a matrix which has: m-horizontal layers (analogous to m-rows), n-vertical page (analogue with n- columns in the usual matrices) and p-vertical layers (p-1 of which are hidden).*

The set of these matrices is written as follows:

$$M_{m\times n\times p}(\mathrm{F}) = \{a_{i,j,k} \mid a_{i,j,k} \in F - field\ \forall i = \overline{1,m}; \quad j = \overline{1,n}; \quad k = \overline{1,p}\}. \tag{1}$$

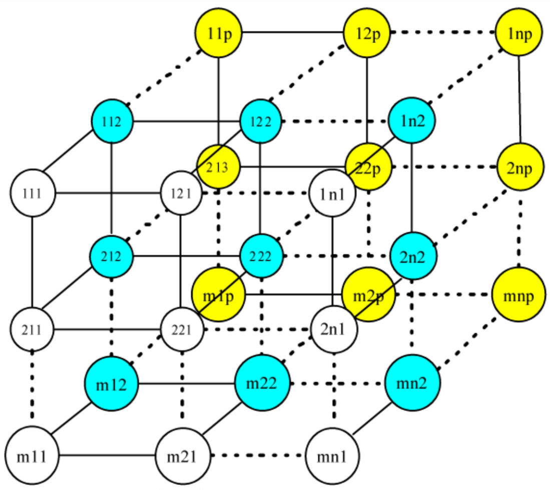

Figure 1. 3D-Matrix view.

The following presents the determinant of 3D-cubic matrices, as well as several properties which are adopted from 2D square determinants.

Cubic-Matrix of Order 2 and 3 and Their Determinants

A 3-dimensional-matrix $A_{n\times n\times n}$ for $n=2,3,\ldots$, called "cubic-matrix of order n". For $n=1$ we have that the cubic-matrix of order 1 is an element of F.

Let us now consider the set of cubic-matrix of order n, for $n=2$ or $n=3$, with elements from a field F (so when cubic-matrix of order n, there are: n −vertical pages, n −horizontal layers and n −vertical layers). From (Zaka 2017, Zaka 2019b) we have that, the addition of 3D-matrix stands also for cubic-matrix of orders 2 and 3. Also, the set of cubic-matrix of order 2 and 3 forms a commutative group (Abelian Group) related to 3Dmatrix addition.

Determinants of Cubic-Matrix of Order 2 and 3

In a paper (Salihu & Zaka 2023b), we have defined and described the meaning of the determinants of cubic-matrix of order 2 and order 3, with elements from a field F. Recall that a cubic-matrix $A_{n\times n\times n}$ for $n=2,3,\ldots$, called "cubic-matrix of order n".

For $n=1$ we have that the cubic-matrix of order 1 is an element of F.
Let us now consider the set of cubic-matrix of order n, with elements from a field F (so when cubic-matrix of order n, there are: n −vertical pages, n −horizontal layers and n −vertical layers),

$$\mathcal{M}_n(F)=\{A_{n\times n\times n}=(a_{ijk})_{n\times n\times n} | a_{ijk}\in F,\ \forall i=\overline{1,n}; j=\overline{1,n}; k=\overline{1,n}\}.$$

In this paper, we define the *determinant of cubic-matrix* as an element from this field, so the map,

$$\begin{aligned}\det: \mathcal{M}_n(F) &\to F\\ \forall A\in\mathcal{M}_n(F) &\mapsto \det(A)\in F.\end{aligned}$$

Below we give two definitions, of how we will calculate the determinant of the cubic-matrix of orders 2 and 3.

Definition 2 *Let $A\in\mathcal{M}_2(F)$ be a $2\times 2\times 2$, with elements from a field F.*

$$A_{2\times2\times2}=\left(\begin{matrix}a_{111} & a_{121}\\ a_{211} & a_{221}\end{matrix}\middle|\begin{matrix}a_{112} & a_{122}\\ a_{212} & a_{222}\end{matrix}\right).$$

The determinant of this cubic-matrix, we called,

$$\det[A_{2\times2\times2}]=\det\left(\begin{matrix}a_{111} & a_{121}\\ a_{211} & a_{221}\end{matrix}\middle|\begin{matrix}a_{112} & a_{122}\\ a_{212} & a_{222}\end{matrix}\right)=a_{111}\cdot a_{222}-a_{112}\cdot a_{221}-a_{121}\cdot a_{212}+a_{122}\cdot a_{211}.$$

The following example is a case where the cubic-matrix, is with elements from the number field $\mathbb{R}$.

Example 1 *Let's have the cubic-matrix, with the element in the number field $\mathbb{R}$,*

$$\det[A_{2\times2\times2}]=\det\left(\begin{matrix}4 & -3\\ -1 & 5\end{matrix}\middle|\begin{matrix}-2 & 4\\ -7 & 3\end{matrix}\right).$$

then according to definition 2, we calculate the Determinant of this cubic-matrix, and have,

$$\det[A_{2\times2\times2}]=\det\left(\begin{matrix}4 & -3\\ -1 & 5\end{matrix}\middle|\begin{matrix}-2 & 4\\ -7 & 3\end{matrix}\right)=4\cdot3-(-2)\cdot5-(-3)\cdot(-7)+4\cdot(-1)$$
$$\det[A_{2\times2\times2}]=12-(-10)-21+(-4)=12+10-21-4=-3.$$

We are trying to expand the meaning of the determinant of cubic-matrix, for order 3 (so when cubic-matrix, there are: 3-vertical pages, 3-horizontal layers and 3-vertical layers).

Definition 3 *Let $A\in\mathcal{M}_3(F)$ be a $3\times3\times3$ cubic-matrix with an element from a field F,*

$$A_{3\times3\times3}=\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right).$$

The determinant of this cubic-matrix, we called,

$$\det[A_{3\times3\times3}]=\det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right). \qquad (2)$$

$$\begin{aligned}\det[A_{3\times3\times3}] = {} & a_{111}\cdot a_{222}\cdot a_{333}-a_{111}\cdot a_{232}\cdot a_{323}-a_{111}\cdot a_{223}\cdot a_{332} \\ & +a_{111}\cdot a_{233}\cdot a_{322}-a_{112}\cdot a_{221}\cdot a_{333}+a_{112}\cdot a_{223}\cdot a_{331} \\ & +a_{112}\cdot a_{231}\cdot a_{323}-a_{112}\cdot a_{233}\cdot a_{321}+a_{113}\cdot a_{221}\cdot a_{332} \\ & -a_{113}\cdot a_{222}\cdot a_{331}-a_{113}\cdot a_{231}\cdot a_{322}+a_{113}\cdot a_{232}\cdot a_{321} \\ & -a_{121}\cdot a_{212}\cdot a_{333}+a_{121}\cdot a_{213}\cdot a_{332}+a_{121}\cdot a_{232}\cdot a_{313} \\ & -a_{121}\cdot a_{233}\cdot a_{312}+a_{122}\cdot a_{211}\cdot a_{333}-a_{122}\cdot a_{213}\cdot a_{331} \\ & -a_{122}\cdot a_{231}\cdot a_{313}+a_{122}\cdot a_{233}\cdot a_{311}-a_{123}\cdot a_{211}\cdot a_{332} \\ & +a_{123}\cdot a_{212}\cdot a_{331}+a_{123}\cdot a_{231}\cdot a_{312}-a_{123}\cdot a_{232}\cdot a_{311} \\ & +a_{131}\cdot a_{212}\cdot a_{323}-a_{131}\cdot a_{213}\cdot a_{322}-a_{131}\cdot a_{222}\cdot a_{313} \\ & +a_{131}\cdot a_{223}\cdot a_{312}-a_{132}\cdot a_{211}\cdot a_{323}+a_{132}\cdot a_{213}\cdot a_{321} \\ & +a_{132}\cdot a_{221}\cdot a_{313}-a_{132}\cdot a_{223}\cdot a_{311}+a_{133}\cdot a_{211}\cdot a_{322} \\ & -a_{133}\cdot a_{212}\cdot a_{321}-a_{133}\cdot a_{221}\cdot a_{312}+a_{133}\cdot a_{222}\cdot a_{311}.\end{aligned}$$

The following example is a case where the cubic-matrix, is with elements from the number field $\mathbb{R}$.

Example 2 *Let's have the cubic-matrix of order 3, with an element from the number field (field of real numbers)* $\mathbb{R}$,

$$\det[A_{3\times3\times3}]=\det\left(\begin{array}{ccc|ccc|ccc} 3 & 0 & -4 & -2 & 4 & 0 & 5 & 1 & 0 \\ 2 & 5 & -1 & -3 & 0 & 3 & 3 & 1 & 2 \\ 0 & 3 & -2 & -3 & 2 & 5 & 0 & 4 & 3 \end{array}\right).$$

Then, we calculate the Determinant of this cubic-matrix following Definition 3, and have that,

$$\det[A_{3\times3\times3}]=\det\left(\begin{array}{ccc|ccc|ccc} 3 & 0 & -4 & -2 & 4 & 0 & 5 & 1 & 0 \\ 2 & 5 & -1 & -3 & 0 & 3 & 3 & 1 & 2 \\ 0 & 3 & -2 & -3 & 2 & 5 & 0 & 4 & 3 \end{array}\right)$$

$$\begin{aligned} & =3\cdot0\cdot3-3\cdot3\cdot4-3\cdot1\cdot5+3\cdot2\cdot2-(-2)5\cdot3+(-2)1(-2)+(-2)(-1)\cdot4-(-2)2\cdot3 \\ & +5\cdot5\cdot5-5\cdot0\cdot(-2)-5\cdot(-1)\cdot2+5\cdot3\cdot3-0\cdot(-3)\cdot3+0\cdot3\cdot5+0\cdot3\cdot0-0\cdot2\cdot(-3) \\ & +4\cdot2\cdot3-4\cdot3\cdot(-2)-4\cdot(-1)\cdot0+4\cdot2\cdot0-1\cdot2\cdot5+1\cdot(-3)\cdot(-2)+1\cdot(-1)\cdot(-3)-1\cdot3\cdot0 \\ & +(-4)(-3)\cdot4-(-4)3\cdot2-(-4)0\cdot0+(-4)1(-3)-0\cdot2\cdot4+0\cdot3\cdot3+0\cdot5\cdot0-0\cdot1\cdot0+ \\ & +0\cdot2\cdot2-0\cdot(-3)\cdot3-0\cdot5\cdot(-3)+0\cdot0\cdot0\end{aligned}$$

so,

$$\begin{aligned}\det[A_{3\times3\times3}] = {} & 0-36-15+12+30+4+8+12+125+0+10+45+0+0+0+0+24+24+0 \\ & +0-10+6+3-0+48+24+0+12-0+0+0-0+0+0+0+0=326.\end{aligned}$$

Hence,

$$\det\left(\begin{array}{ccc|ccc|ccc} 3 & 0 & -4 & -2 & 4 & 0 & 5 & 1 & 0 \\ 2 & 5 & -1 & -3 & 0 & 3 & 3 & 1 & 2 \\ 0 & 3 & -2 & -3 & 2 & 5 & 0 & 4 & 3 \end{array}\right)=326.$$

Minors and Co-factors of Cubic-Matrix of Order 2 and 3

In this section, we will present the meaning of Minors and co-factors for the cubic-matrix of order 2 and order 3.

Minors of Cubic-Matrix

Let us start by defining minors.

Definition 4 *Let A_n be a $n \times n \times n$ cubic-matrix (with $n \geq 2$). Denote by A_{ijk} the entry of cubic-matrix A at the intersection of the i −th* ***horizontal layers****, j −th* ***vertical pages*** *and k −th* ***vertical layers****. The minor of A_{ijk} is the determinant of the sub-cubic-matrix obtained from A by deleting its i −th* ***horizontal layer****, j* ***−vertical page*** *and k* ***−vertical layer****.*

We now illustrate the definition with an example.

Example 3 *Let's have the cubic-matrix of order 3, with an element from the number field (field of real numbers) $\mathbb{R}$,*

$$A_{3\times3\times3} = \left(\begin{array}{ccc|ccc|ccc} 3 & 0 & -4 & -2 & 4 & 0 & 5 & 1 & 0 \\ 2 & 5 & -1 & -3 & 0 & 3 & 3 & 1 & 2 \\ 0 & 3 & -2 & -3 & 2 & 5 & 0 & 4 & 3 \end{array}\right).$$

Take the entry $A_{111} = 3$, The sub-cubic-matrix obtained by deleting the first-horizontal layer, first-vertical page and first-vertical layer is,

$$\left(\begin{array}{cc|cc} 0 & 3 & 1 & 2 \\ 2 & 5 & 4 & 3 \end{array}\right).$$

Thus, the minor of A_{111} is

$$M_{111} = \det\left(\begin{array}{cc|cc} 0 & 3 & 1 & 2 \\ 2 & 5 & 4 & 3 \end{array}\right) = 0\cdot3 - 1\cdot5 - 3\cdot4 + 2\cdot2 = -5 - 12 + 4 = -13.$$

Take the entry $A_{123} = 1$, The sub-cubic-matrix obtained by deleting the first-horizontal layer, 2-vertical page and 3-vertical layer is,

$$\left(\begin{array}{cc|cc} 2 & -1 & -3 & 3 \\ 0 & -2 & -3 & 5 \end{array}\right).$$

Thus, the minor of A_{123} is

$$M_{123} = \det\left(\begin{array}{cc|cc} 2 & -1 & -3 & 3 \\ 0 & -2 & -3 & 5 \end{array}\right) = 2\cdot5 - (-3)\cdot(-2) - (-1)\cdot(-3) + 3\cdot0 = 10 - 6 - 3 + 0 = 1.$$

Co-Factors of Cubic-Matrix of Order 2 and 3

A co-factor is a minor whose sign may have been changed depending on the location of the respective matrix entry.

Definition 5 Let A_n be a $n \times n \times n$ cubic-matrix (with $n \geq 2$). Denote by M_{ijk} the minor of an entry A_{ijk}. The co-factor of A_{ijk} is

$$C_{ijk} = (-1)^{i+j+k} \cdot M_{ijk}.$$

As an example, the pattern of sign changes $(-1)^{i+j+k}$ of a cubic-matrix of order 3 is

$$\left(\begin{array}{ccc|ccc|ccc} - & + & - & + & - & + & - & + & - \\ + & - & + & - & + & - & + & - & + \\ - & + & - & + & - & + & - & + & - \end{array}\right).$$

Example 4 *Let's have the cubic-matrix of order 3, with an element from the number field (field of real numbers) $\mathbb{R}$,*

$$A_{3\times3\times3} = \left(\begin{matrix}3 & 0 & -4\\ 2 & 5 & -1\\ 0 & 3 & -2\end{matrix}\middle|\begin{matrix}-2 & 4 & 0\\ -3 & 0 & 3\\ -3 & 2 & 5\end{matrix}\middle|\begin{matrix}5 & 1 & 0\\ 3 & 1 & 2\\ 0 & 4 & 3\end{matrix}\right).$$

Take the entry $A_{111} = 3$. The minor of A_{111} is

$$M_{111} = \det\left(\begin{matrix}0 & 3\\ 2 & 5\end{matrix}\middle|\begin{matrix}1 & 2\\ 4 & 3\end{matrix}\right) = -13$$

and its cofactor is

$$C_{111} = (-1)^{1+1+1} \cdot M_{111} = -M_{111} = -(-13) = 13.$$

Take the entry $A_{123} = 1$. Thus, the minor of A_{123} is

$$M_{123} = \det\left(\begin{matrix}2 & -1\\ 0 & -2\end{matrix}\middle|\begin{matrix}-3 & 3\\ -3 & 5\end{matrix}\right) = 1$$

and its co-factor is

$$C_{123} = (-1)^{1+2+3} \cdot M_{123} = M_{123} = 1.$$

Laplace Expansion for Determinants of Cubic-Matrix of Order 2 and 3

We are now ready to present the Laplace expansion. Following the Laplace expansion method for 2D square-matrix, we are conjecturing this method for 3D cubic-matrix,

Laplace Expansion for Determinants of Cubic-Matrix of Order 2

Laplace Expansion

If we have A a cubic-matrix of order 2 or 3. Denote by C_{ijk} the co-factor of an entry A_{ijk}. Then:

For any '***horizontal laye***r' i, the following 'horizontal layer' expansion holds:

$$\det(A) = \sum_{jk} A_{ijk} \cdot C_{ijk}.$$

For any '***vertical page***' j, the following 'vertical page' expansion holds:

$$\det(A) = \sum_{ik} A_{ijk} \cdot C_{ijk}.$$

For any '***vertical layer***' k, the following 'vertical page' expansion holds:

$$\det(A) = \sum_{ij} A_{ijk} \cdot C_{ijk}.$$

Below we prove that this method is valid for calculating the determinants of the cubic-matrix of order 2.

Theorem 1 *Let A be a cubic-matrix of order 2,*

$$A = \left(\begin{matrix}a_{111} & a_{121}\\ a_{211} & a_{221}\end{matrix}\middle|\begin{matrix}a_{112} & a_{122}\\ a_{212} & a_{222}\end{matrix}\right).$$

The determinant of this cubic-matrix is invariant into the expansion of three "ways" to Laplace expansion.
Proof. We will prove all three expansion types, L_1, L_2, L_3.
(L_1): For any horizontal layer i ($i = 1,2$), the following 'horizontal layer' expansion holds:

$$\det(A) = \sum_{jk} A_{ijk} \cdot C_{ijk}.$$

If we take $i = 1$, and we consider the meaning of the minors and co-factors for the cubic-matrix of order 2, which we described above, we have:

$$\det[A_{2\times2\times2}] = \begin{pmatrix} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{pmatrix}$$
$$= a_{111} \cdot \det(a_{222}) - a_{121} \cdot \det(a_{212}) - a_{112} \cdot \det(a_{221}) + a_{122} \cdot \det(a_{211})$$

so,

$$\det[A_{2\times2\times2}] = a_{111} \cdot a_{222} - a_{112} \cdot a_{221} - a_{121} \cdot a_{212} + a_{122} \cdot a_{211}. \quad (3)$$

This result is the same as that in the Definition 2.
Now similarly we take $i = 2$, and we consider the meaning of the minors and co-factors for the cubic-matrix of order 2, which we described above, we have:

$$\begin{aligned} \det[A_{2\times2\times2}] &= \begin{pmatrix} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{pmatrix} \\ &= a_{211} \cdot \det(a_{122}) - a_{221} \cdot \det(a_{112}) - a_{212} \cdot \det(a_{121}) + a_{222} \cdot \det(a_{111}) \\ &= a_{211} \cdot a_{122} - a_{221} \cdot a_{112} - a_{212} \cdot a_{121} + a_{222} \cdot a_{111}. \\ so, & \\ \det[A_{2\times2\times2}] &= a_{211} \cdot a_{122} - a_{221} \cdot a_{112} - a_{212} \cdot a_{121} + a_{222} \cdot a_{111}. \end{aligned} \quad (4)$$

We see that we have the same result as the Definition 2.
(L_2): for any 'vertical page' j , the following 'vertical page' expansion holds:
We see that we have the same result as the Definition 2.
(L_2): for any 'vertical page' j , the following 'vertical page' expansion holds:

$$\det(A) = \sum_{ik} A_{ijk} \cdot C_{ijk}.$$

If we take $j = 1$, and we consider the meaning of the minors and co-factors for the cubic-matrix of order 2, which we described above, we have:

$$\begin{aligned} \det[A_{2\times2\times2}] &= \begin{pmatrix} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{pmatrix} \\ &= a_{111} \cdot \det(a_{222}) + a_{211} \cdot \det(a_{122}) \\ &-a_{112} \cdot \det(a_{221}) - a_{212} \cdot \det(a_{121}) \\ &= a_{111} \cdot a_{222} + a_{211} \cdot a_{122} \\ &-a_{112} \cdot a_{221} + a_{212} \cdot a_{121}. \end{aligned} \quad (5)$$

So we have the same result as the Definition 2.
Now similarly we take $j = 2$, and we consider the meaning of the minors and co-factors for the cubic-matrix of order 2, which we described above, we have:

$$\begin{aligned} \det[A_{2\times2\times2}] &= \begin{pmatrix} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{pmatrix} \\ &= -a_{121} \cdot \det(a_{212}) - a_{221} \cdot \det(a_{112}) \\ &+a_{122} \cdot \det(a_{211}) + a_{222} \cdot \det(a_{111}) \\ &= -a_{121} \cdot a_{212} - a_{221} \cdot a_{112} \\ &+a_{122} \cdot a_{211} + a_{222} \cdot a_{111}. \end{aligned} \quad (6)$$

So we have the same result as the Definition 2.

(L_3): For any 'vertical layer' k , the following 'vertical layer' expansion holds:

$$\det(A) = \sum_{ik} A_{ijk} \cdot C_{ijk}.$$

If we take $k = 1$, and we consider the meaning of the minors and co-factors for the cubic-matrix of order 2, which we described above, we have:

$$\begin{aligned} \det[A_{2\times2\times2}] &= \begin{pmatrix} a_{111} & a_{121} \\ a_{211} & a_{221} \end{pmatrix}\left|\begin{matrix} a_{112} & a_{122} \\ a_{212} & a_{222} \end{matrix}\right) \\ &= a_{111} \cdot \det(a_{222}) - a_{121} \cdot \det(a_{212}) \\ &+ a_{211} \cdot \det(a_{122}) - a_{221} \cdot \det(a_{112}) \\ &= a_{111} \cdot a_{222} - a_{121} \cdot a_{212} \\ &+ a_{211} \cdot a_{122} - a_{221} \cdot a_{112}. \end{aligned} \tag{7}$$

So we have the same result as the Definition 2.
Now similarly we take $k = 2$, and we consider the meaning of the minors and co-factors for the cubic-matrix of order 2, which we described above, we have:

$$\begin{aligned} \det[A_{2\times2\times2}] &= \begin{pmatrix} a_{111} & a_{121} \\ a_{211} & a_{221} \end{pmatrix}\left|\begin{matrix} a_{112} & a_{122} \\ a_{212} & a_{222} \end{matrix}\right) \\ &= -a_{112} \cdot \det(a_{221}) + a_{122} \cdot \det(a_{221}) \\ &- a_{212} \cdot \det(a_{121}) + a_{222} \cdot \det(a_{111}) \\ &= -a_{112} \cdot a_{221} + a_{122} \cdot a_{221} \\ &- a_{212} \cdot a_{121} + a_{222} \cdot a_{111}. \end{aligned} \tag{8}$$

So we have the same result as the Definition 2.
The following example is a case where the cubic-matrix of second order, is with elements from the number field $\mathbb{R}$.

Example 5 *Let's have the cubic-matrix, with the element in the number field* $\mathbb{R}$,

$$A_{2\times2\times2} = \begin{pmatrix} 4 & -3 \\ -1 & 5 \end{pmatrix}\left|\begin{matrix} -2 & 4 \\ -7 & 3 \end{matrix}\right)$$

then according to Theorem 1, we calculate the Determinant of this cubic-matrix, and have,

$$\det[A_{2\times2\times2}] = \det \begin{pmatrix} 4 & -3 \\ -1 & 5 \end{pmatrix}\left|\begin{matrix} -2 & 4 \\ -7 & 3 \end{matrix}\right).$$

For $i = 1$, we have:

$$\begin{aligned} \det[A_{2\times2\times2}] &= \begin{pmatrix} 4 & -3 \\ -1 & 5 \end{pmatrix}\left|\begin{matrix} -2 & 4 \\ -7 & 3 \end{matrix}\right) \\ &= 4 \cdot \det(3) - (-3) \cdot \det(-7) \\ &- (-2) \cdot \det(5) + 4 \cdot \det(-1) \\ &= 4 \cdot 3 - (-3) \cdot (-7) - (-2) \cdot 5 \\ &+ 4 \cdot (-1) = -3. \end{aligned}$$

We see that we have the same result as the Example 1.
For $i = 2$, we have:

$$\begin{aligned} \det[A_{2\times2\times2}] &= \begin{pmatrix} 4 & -3 \\ -1 & 5 \end{pmatrix}\left|\begin{matrix} -2 & 4 \\ -7 & 3 \end{matrix}\right) \\ &= -1 \cdot \det(4) - 5 \cdot \det(-2) \\ &- (-7) \cdot \det(-3) + 3 \cdot \det(4) \\ &= -1 \cdot 4 - 5 \cdot (-2) - (-7) \cdot (-3) \\ &+ 3 \cdot 4 = -3. \end{aligned}$$

So we get the same result as the Example 1.
For $j=1$, we have:

$$\begin{aligned}\det[A_{2\times2\times2}] &= \left(\begin{matrix}4 & -3\\ -1 & 5\end{matrix}\middle|\begin{matrix}-2 & 4\\ -7 & 3\end{matrix}\right)\\ &= 4\cdot\det(3)+(-1)\cdot\det(4)\\ &-(-2)\cdot\det(5)-(-7)\cdot\det(-3)\\ &= 4\cdot3+(-1)\cdot4-(-2)\cdot5\\ &-(-7)\cdot(-3)=-3.\end{aligned}$$

So we get the same result as the Example 1.
For $j=2$, we have:

$$\begin{aligned}\det[A_{2\times2\times2}] &= \left(\begin{matrix}4 & -3\\ -1 & 5\end{matrix}\middle|\begin{matrix}-2 & 4\\ -7 & 3\end{matrix}\right)\\ &= -(-3)\cdot\det(-7)-5\cdot\det(-2)\\ &+4\cdot\det(-1)+3\cdot\det(4)\\ &= 3\cdot(-7)-5\cdot(-2)-4\cdot(-1)\\ &+3\cdot4=-3.\end{aligned}$$

So we get the same result as the Example 1.
For $k=1$, we have:

$$\begin{aligned}\det[A_{2\times2\times2}] &= \left(\begin{matrix}4 & -3\\ -1 & 5\end{matrix}\middle|\begin{matrix}-2 & 4\\ -7 & 3\end{matrix}\right)\\ &= 4\cdot\det(3)-(-3)\cdot\det(-7)\\ &+(-1)\cdot\det(4)-5\cdot\det(-2)\\ &= 4\cdot3-(-3)\cdot(-7)+(-1)\cdot4\\ &-5\cdot(-2)=-3.\end{aligned}$$

So we have the same result as the Example 1.
For $k=2$, we have:

$$\begin{aligned}\det[A_{2\times2\times2}] &= \left(\begin{matrix}4 & -3\\ -1 & 5\end{matrix}\middle|\begin{matrix}-2 & 4\\ -7 & 3\end{matrix}\right)\\ &= -(-2)\cdot\det(5)+4\cdot\det(5)\\ &-(-7)\cdot\det(-3)+3\cdot\det(4)\\ &= -(-2)\cdot5+4\cdot5-(-7)\cdot(-3)\\ &+3\cdot4=-3.\end{aligned}$$

So we have the same result as the Example 1.

Laplace Expansion for Determinants Of Cubic-Matrix of Order 3

Below we prove that this method is valid for calculating the determinants of the cubic-matrix of order 3.

Theorem 2 *Let A be a cubic-matrix of order 3,*

$$A=\left(\begin{matrix}a_{111} & a_{121} & a_{131}\\ a_{211} & a_{221} & a_{231}\\ a_{311} & a_{321} & a_{331}\end{matrix}\middle|\begin{matrix}a_{112} & a_{122} & a_{132}\\ a_{212} & a_{222} & a_{232}\\ a_{312} & a_{322} & a_{332}\end{matrix}\middle|\begin{matrix}a_{113} & a_{123} & a_{133}\\ a_{213} & a_{223} & a_{233}\\ a_{313} & a_{323} & a_{333}\end{matrix}\right).$$

The determinant of this cubic-matrix is invariant into the expansion of three "ways" to Laplace expansion.
Proof. We will prove all three expansion types, L_1, L_2, L_3 also for the third order.
(L_1): For any horizontal layer i $(i=1,2,3)$, the following 'horizontal layer' expansion holds:

$$\det(A)=\sum_{jk} A_{ijk}\cdot C_{ijk}.$$

If we take $i = 1$, and we consider the meaning of the minors and co-factors for the cubic-matrix of order 3, which we described above, we have:

$$\det(A) = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$= a_{111} \cdot \left(\begin{array}{cc|cc} a_{222} & a_{232} & a_{223} & a_{233} \\ a_{322} & a_{332} & a_{323} & a_{333} \end{array}\right) - a_{121} \cdot \left(\begin{array}{cc|cc} a_{212} & a_{232} & a_{213} & a_{233} \\ a_{312} & a_{332} & a_{313} & a_{333} \end{array}\right) + a_{131} \cdot \left(\begin{array}{cc|cc} a_{212} & a_{222} & a_{213} & a_{223} \\ a_{312} & a_{322} & a_{313} & a_{323} \end{array}\right)$$
$$- a_{112} \cdot \left(\begin{array}{cc|cc} a_{221} & a_{231} & a_{223} & a_{233} \\ a_{321} & a_{331} & a_{323} & a_{333} \end{array}\right) + a_{122} \cdot \left(\begin{array}{cc|cc} a_{211} & a_{231} & a_{213} & a_{233} \\ a_{311} & a_{331} & a_{313} & a_{333} \end{array}\right) - a_{132} \cdot \left(\begin{array}{cc|cc} a_{211} & a_{221} & a_{213} & a_{223} \\ a_{311} & a_{321} & a_{313} & a_{323} \end{array}\right)$$
$$+ a_{113} \cdot \left(\begin{array}{cc|cc} a_{221} & a_{231} & a_{222} & a_{232} \\ a_{321} & a_{331} & a_{322} & a_{332} \end{array}\right) - a_{123} \cdot \left(\begin{array}{cc|cc} a_{211} & a_{231} & a_{212} & a_{232} \\ a_{311} & a_{331} & a_{312} & a_{332} \end{array}\right) + a_{133} \cdot \left(\begin{array}{cc|cc} a_{211} & a_{221} & a_{212} & a_{222} \\ a_{311} & a_{321} & a_{312} & a_{322} \end{array}\right).$$

After expanding further the above determinant based on Theorem 1, we see that this result is the same as the result of Definition 3.

If we take $i = 2$, and we consider the meaning of the minors and co-factors for the cubic-matrix of order 3, which we described above, we have:

$$\det(A) = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$= a_{211} \cdot \left(\begin{array}{cc|cc} a_{122} & a_{132} & a_{123} & a_{133} \\ a_{322} & a_{332} & a_{323} & a_{333} \end{array}\right) - a_{221} \cdot \left(\begin{array}{cc|cc} a_{112} & a_{132} & a_{113} & a_{133} \\ a_{312} & a_{332} & a_{313} & a_{333} \end{array}\right) + a_{231} \cdot \left(\begin{array}{cc|cc} a_{112} & a_{122} & a_{113} & a_{123} \\ a_{312} & a_{322} & a_{313} & a_{323} \end{array}\right)$$
$$- a_{212} \cdot \left(\begin{array}{cc|cc} a_{121} & a_{131} & a_{123} & a_{133} \\ a_{321} & a_{331} & a_{323} & a_{333} \end{array}\right) + a_{222} \cdot \left(\begin{array}{cc|cc} a_{111} & a_{131} & a_{113} & a_{133} \\ a_{311} & a_{331} & a_{313} & a_{333} \end{array}\right) - a_{232} \cdot \left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{113} & a_{123} \\ a_{311} & a_{321} & a_{313} & a_{323} \end{array}\right)$$
$$+ a_{113} \cdot \left(\begin{array}{cc|cc} a_{221} & a_{231} & a_{222} & a_{232} \\ a_{321} & a_{331} & a_{322} & a_{332} \end{array}\right) - a_{123} \cdot \left(\begin{array}{cc|cc} a_{211} & a_{231} & a_{212} & a_{232} \\ a_{311} & a_{331} & a_{312} & a_{332} \end{array}\right) + a_{133} \cdot \left(\begin{array}{cc|cc} a_{211} & a_{221} & a_{212} & a_{222} \\ a_{311} & a_{321} & a_{312} & a_{322} \end{array}\right).$$

After expanding further the above determinant based on Theorem 1, we see that this result is the same as the result of Definition 3.
If we take $i = 3$, and we consider the meaning of the minors and co-factors for the cubic-matrix of order 3, which we described above, we have:

$$\det(A) = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$= a_{311} \cdot \left(\begin{array}{cc|cc} a_{122} & a_{132} & a_{123} & a_{133} \\ a_{222} & a_{232} & a_{223} & a_{233} \end{array}\right) - a_{221} \cdot \left(\begin{array}{cc|cc} a_{112} & a_{132} & a_{113} & a_{133} \\ a_{212} & a_{232} & a_{213} & a_{233} \end{array}\right) + a_{331} \cdot \left(\begin{array}{cc|cc} a_{112} & a_{122} & a_{113} & a_{123} \\ a_{212} & a_{222} & a_{213} & a_{223} \end{array}\right)$$
$$- a_{312} \cdot \left(\begin{array}{cc|cc} a_{121} & a_{131} & a_{123} & a_{133} \\ a_{221} & a_{231} & a_{223} & a_{233} \end{array}\right) + a_{322} \cdot \left(\begin{array}{cc|cc} a_{111} & a_{131} & a_{113} & a_{133} \\ a_{211} & a_{231} & a_{213} & a_{233} \end{array}\right) - a_{332} \cdot \left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{113} & a_{123} \\ a_{211} & a_{221} & a_{213} & a_{223} \end{array}\right)$$
$$+ a_{313} \cdot \left(\begin{array}{cc|cc} a_{221} & a_{231} & a_{222} & a_{232} \\ a_{221} & a_{231} & a_{222} & a_{232} \end{array}\right) - a_{323} \cdot \left(\begin{array}{cc|cc} a_{211} & a_{231} & a_{212} & a_{232} \\ a_{211} & a_{231} & a_{212} & a_{232} \end{array}\right) + a_{333} \cdot \left(\begin{array}{cc|cc} a_{211} & a_{221} & a_{212} & a_{222} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right).$$

After expanding further the above determinant based on Theorem 1, we see that this result is the same as the result of Definition 3.

If we take $j = 1$, and we consider the meaning of the minors and co-factors for the cubic-matrix of order 3, which we described above, we have:

$$\det(A) = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$= a_{111} \cdot \left(\begin{array}{cc|cc} a_{222} & a_{232} & a_{223} & a_{233} \\ a_{322} & a_{332} & a_{323} & a_{333} \end{array}\right) - a_{211} \cdot \left(\begin{array}{cc|cc} a_{122} & a_{132} & a_{123} & a_{133} \\ a_{322} & a_{332} & a_{323} & a_{333} \end{array}\right) + a_{311} \cdot \left(\begin{array}{cc|cc} a_{122} & a_{132} & a_{123} & a_{133} \\ a_{222} & a_{232} & a_{223} & a_{233} \end{array}\right)$$
$$- a_{112} \cdot \left(\begin{array}{cc|cc} a_{221} & a_{231} & a_{223} & a_{233} \\ a_{321} & a_{331} & a_{323} & a_{333} \end{array}\right) + a_{212} \cdot \left(\begin{array}{cc|cc} a_{121} & a_{131} & a_{123} & a_{133} \\ a_{321} & a_{331} & a_{323} & a_{333} \end{array}\right) - a_{312} \cdot \left(\begin{array}{cc|cc} a_{121} & a_{131} & a_{123} & a_{133} \\ a_{221} & a_{231} & a_{223} & a_{233} \end{array}\right)$$
$$+ a_{113} \cdot \left(\begin{array}{cc|cc} a_{221} & a_{231} & a_{222} & a_{232} \\ a_{321} & a_{331} & a_{322} & a_{332} \end{array}\right) - a_{213} \cdot \left(\begin{array}{cc|cc} a_{121} & a_{131} & a_{122} & a_{132} \\ a_{321} & a_{331} & a_{322} & a_{332} \end{array}\right) + a_{313} \cdot \left(\begin{array}{cc|cc} a_{121} & a_{131} & a_{122} & a_{132} \\ a_{221} & a_{231} & a_{222} & a_{232} \end{array}\right).$$

After expanding further the above determinant based on Theorem 1, we see that this result is the same as the result of Definition 3.

If we take $j = 2$, and we consider the meaning of the minors and co-factors for the cubic-matrix of order 3, which we described above, we have:

$$\det(A) = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$= a_{121}\cdot\left(\begin{array}{cc|cc} a_{212} & a_{232} & a_{213} & a_{233} \\ a_{312} & a_{332} & a_{313} & a_{333} \end{array}\right) - a_{221}\cdot\left(\begin{array}{cc|cc} a_{112} & a_{132} & a_{113} & a_{133} \\ a_{312} & a_{332} & a_{313} & a_{333} \end{array}\right) + a_{321}\cdot\left(\begin{array}{cc|cc} a_{112} & a_{132} & a_{113} & a_{133} \\ a_{212} & a_{232} & a_{213} & a_{233} \end{array}\right)$$
$$-a_{122}\cdot\left(\begin{array}{cc|cc} a_{211} & a_{231} & a_{213} & a_{233} \\ a_{311} & a_{331} & a_{313} & a_{333} \end{array}\right) + a_{222}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{131} & a_{113} & a_{133} \\ a_{311} & a_{331} & a_{313} & a_{333} \end{array}\right) - a_{322}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{131} & a_{113} & a_{133} \\ a_{211} & a_{231} & a_{213} & a_{233} \end{array}\right)$$
$$+a_{123}\cdot\left(\begin{array}{cc|cc} a_{211} & a_{231} & a_{212} & a_{232} \\ a_{311} & a_{331} & a_{312} & a_{332} \end{array}\right) - a_{223}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{131} & a_{112} & a_{132} \\ a_{311} & a_{331} & a_{312} & a_{332} \end{array}\right) + a_{323}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{131} & a_{112} & a_{132} \\ a_{211} & a_{231} & a_{212} & a_{232} \end{array}\right)$$

After expanding further the above determinant based on Theorem 1, we see that this result is the same as the result of Definition 3.

If we take $j = 3$, and we consider the meaning of the minors and co-factors for the cubic-matrix of order 3, which we described above, we have:

$$\det(A) = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$= a_{131}\cdot\left(\begin{array}{cc|cc} a_{212} & a_{222} & a_{213} & a_{223} \\ a_{312} & a_{322} & a_{313} & a_{323} \end{array}\right) - a_{231}\cdot\left(\begin{array}{cc|cc} a_{112} & a_{122} & a_{113} & a_{123} \\ a_{312} & a_{322} & a_{313} & a_{323} \end{array}\right) + a_{331}\cdot\left(\begin{array}{cc|cc} a_{112} & a_{122} & a_{113} & a_{123} \\ a_{212} & a_{222} & a_{213} & a_{223} \end{array}\right)$$
$$-a_{132}\cdot\left(\begin{array}{cc|cc} a_{211} & a_{221} & a_{213} & a_{223} \\ a_{311} & a_{321} & a_{313} & a_{323} \end{array}\right) + a_{232}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{113} & a_{123} \\ a_{311} & a_{321} & a_{313} & a_{323} \end{array}\right) - a_{332}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{113} & a_{123} \\ a_{211} & a_{221} & a_{213} & a_{223} \end{array}\right)$$
$$+a_{133}\cdot\left(\begin{array}{cc|cc} a_{211} & a_{221} & a_{212} & a_{222} \\ a_{311} & a_{321} & a_{312} & a_{322} \end{array}\right) - a_{233}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{311} & a_{321} & a_{312} & a_{322} \end{array}\right) + a_{333}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right).$$

After expanding further the above determinant based on Theorem 1, we see that this result is the same as the result of Definition 3.

If we take $k = 1$, and we consider the meaning of the minors and co-factors for the cubic-matrix of order 3, which we described above, we have:

$$\det(A) = \det\left[\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)\right]$$

$$= a_{111}\cdot\left(\begin{array}{cc|cc} a_{222} & a_{232} & a_{223} & a_{233} \\ a_{322} & a_{332} & a_{323} & a_{333} \end{array}\right) - a_{121}\cdot\left(\begin{array}{cc|cc} a_{212} & a_{232} & a_{213} & a_{233} \\ a_{312} & a_{332} & a_{313} & a_{333} \end{array}\right) + a_{131}\cdot\left(\begin{array}{cc|cc} a_{212} & a_{222} & a_{213} & a_{223} \\ a_{312} & a_{322} & a_{313} & a_{323} \end{array}\right)$$
$$-a_{211}\cdot\left(\begin{array}{cc|cc} a_{122} & a_{132} & a_{123} & a_{133} \\ a_{322} & a_{332} & a_{323} & a_{333} \end{array}\right) + a_{221}\cdot\left(\begin{array}{cc|cc} a_{112} & a_{132} & a_{113} & a_{133} \\ a_{312} & a_{332} & a_{313} & a_{333} \end{array}\right) - a_{231}\cdot\left(\begin{array}{cc|cc} a_{112} & a_{122} & a_{113} & a_{123} \\ a_{312} & a_{322} & a_{313} & a_{323} \end{array}\right)$$
$$+a_{311}\cdot\left(\begin{array}{cc|cc} a_{122} & a_{132} & a_{223} & a_{233} \\ a_{222} & a_{232} & a_{323} & a_{333} \end{array}\right) - a_{321}\cdot\left(\begin{array}{cc|cc} a_{112} & a_{132} & a_{123} & a_{133} \\ a_{212} & a_{232} & a_{213} & a_{233} \end{array}\right) + a_{331}\cdot\left(\begin{array}{cc|cc} a_{112} & a_{122} & a_{113} & a_{123} \\ a_{212} & a_{222} & a_{213} & a_{223} \end{array}\right).$$

After expanding further the above determinant based on Theorem 1, we see that this result is the same as the result of Definition 3.

If we take $k = 2$, and we consider the meaning of the minors and co-factors for the cubic-matrix of order 3, which we described above, we have:

$$\det(A) = \det\left[\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)\right]$$

$$= a_{112}\cdot\left(\begin{array}{cc|cc} a_{221} & a_{231} & a_{223} & a_{233} \\ a_{321} & a_{331} & a_{323} & a_{333} \end{array}\right) - a_{122}\cdot\left(\begin{array}{cc|cc} a_{211} & a_{231} & a_{213} & a_{233} \\ a_{311} & a_{331} & a_{313} & a_{333} \end{array}\right) + a_{132}\cdot\left(\begin{array}{cc|cc} a_{211} & a_{221} & a_{213} & a_{223} \\ a_{311} & a_{321} & a_{313} & a_{323} \end{array}\right)$$
$$-a_{212}\cdot\left(\begin{array}{cc|cc} a_{121} & a_{131} & a_{123} & a_{133} \\ a_{321} & a_{331} & a_{323} & a_{333} \end{array}\right) + a_{222}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{131} & a_{113} & a_{133} \\ a_{311} & a_{331} & a_{313} & a_{333} \end{array}\right) - a_{232}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{113} & a_{123} \\ a_{311} & a_{321} & a_{313} & a_{323} \end{array}\right)$$

$$+a_{312}\cdot\left(\begin{array}{cc|cc} a_{121} & a_{131} & a_{223} & a_{233} \\ a_{221} & a_{231} & a_{323} & a_{333} \end{array}\right) - a_{322}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{131} & a_{123} & a_{133} \\ a_{211} & a_{231} & a_{213} & a_{233} \end{array}\right) + a_{332}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{113} & a_{123} \\ a_{211} & a_{221} & a_{213} & a_{223} \end{array}\right).$$

After expanding further the above determinant based on Theorem 1, we see that this result is the same as the result of Definition 3.

If we take $k = 3$, and we consider the meaning of the minors and co-factors for the cubic-matrix of order 3, which we described above, we have:

$$\det(A) = \det\left[\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)\right]$$

$$= a_{113}\cdot\left(\begin{array}{cc|cc} a_{221} & a_{231} & a_{222} & a_{232} \\ a_{321} & a_{331} & a_{322} & a_{332} \end{array}\right) - a_{123}\cdot\left(\begin{array}{cc|cc} a_{211} & a_{231} & a_{212} & a_{232} \\ a_{311} & a_{331} & a_{312} & a_{332} \end{array}\right) + a_{133}\cdot\left(\begin{array}{cc|cc} a_{211} & a_{221} & a_{212} & a_{222} \\ a_{311} & a_{321} & a_{312} & a_{322} \end{array}\right)$$

$$-a_{213}\cdot\left(\begin{array}{cc|cc} a_{121} & a_{131} & a_{122} & a_{132} \\ a_{321} & a_{331} & a_{322} & a_{332} \end{array}\right) + a_{223}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{131} & a_{112} & a_{132} \\ a_{311} & a_{331} & a_{312} & a_{332} \end{array}\right) - a_{233}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{311} & a_{321} & a_{312} & a_{322} \end{array}\right)$$

$$+a_{313}\cdot\left(\begin{array}{cc|cc} a_{121} & a_{131} & a_{222} & a_{232} \\ a_{221} & a_{231} & a_{322} & a_{332} \end{array}\right) - a_{323}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{131} & a_{122} & a_{132} \\ a_{211} & a_{231} & a_{212} & a_{232} \end{array}\right) + a_{333}\cdot\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right).$$

After expanding further the above determinant based on Theorem 1, we see that this result is the same as the result of Definition 3.

The following example is a case where the cubic-matrix of the third order, is with elements from the number field $\mathbb{R}$.

Example 6 *Let's have the cubic-matrix, with the element in the number field* $\mathbb{R}$,

$$A_{3\times3\times3} = \left(\begin{array}{ccc|ccc|ccc} 3 & 0 & -4 & -2 & 4 & 0 & 5 & 1 & 0 \\ 2 & 5 & -1 & -3 & 0 & 3 & 3 & 1 & 2 \\ 0 & 3 & -2 & -3 & 2 & 5 & 0 & 4 & 3 \end{array}\right)$$

then according to Theorem 2, we calculate the Determinant of this cubic-matrix, and have,

$$\det[A_{3\times3\times3}] = \det\left[\left(\begin{array}{ccc|ccc|ccc} 3 & 0 & -4 & -2 & 4 & 0 & 5 & 1 & 0 \\ 2 & 5 & -1 & -3 & 0 & 3 & 3 & 1 & 2 \\ 0 & 3 & -2 & -3 & 2 & 5 & 0 & 4 & 3 \end{array}\right)\right].$$

For $i = 1$, we have:

$$\det(A) = \det\left[\left(\begin{array}{ccc|ccc|ccc} 3 & 0 & -4 & -2 & 4 & 0 & 5 & 1 & 0 \\ 2 & 5 & -1 & -3 & 0 & 3 & 3 & 1 & 2 \\ 0 & 3 & -2 & -3 & 2 & 5 & 0 & 4 & 3 \end{array}\right)\right]$$

$$= 3\cdot\left(\begin{array}{cc|cc} 0 & 3 & 1 & 2 \\ 2 & 5 & 4 & 3 \end{array}\right) - 0\cdot\left(\begin{array}{cc|cc} -3 & 3 & 3 & 2 \\ -3 & 5 & 0 & 3 \end{array}\right) + (-4)\cdot\left(\begin{array}{cc|cc} -3 & 0 & 3 & 1 \\ -3 & 2 & 0 & 4 \end{array}\right) - (-2)\cdot\left(\begin{array}{cc|cc} 5 & -1 & 1 & 2 \\ 3 & -2 & 4 & 3 \end{array}\right)$$

$$+4\cdot\left(\begin{array}{cc|cc} 2 & -1 & 3 & 2 \\ 0 & -2 & 0 & 3 \end{array}\right) - 0\cdot\left(\begin{array}{cc|cc} 2 & 5 & 3 & 1 \\ 0 & 3 & 0 & 4 \end{array}\right) + 5\cdot\left(\begin{array}{cc|cc} 5 & -1 & 0 & 3 \\ 3 & -2 & 2 & 5 \end{array}\right) - 1\cdot\left(\begin{array}{cc|cc} 2 & -1 & -3 & 3 \\ 0 & -2 & -3 & 5 \end{array}\right) + 0\cdot\left(\begin{array}{cc|cc} 2 & 5 & -3 & 0 \\ 0 & 3 & -3 & 2 \end{array}\right)$$

So

$$\det(\mathrm{A}) = 326.$$

After expanding further the minors of the above determinant based on Theorem 1, we see that this result is the same as the result of Example 2.
For $i = 2$, we have:

$$\det(A) = \det\left[\left(\begin{array}{ccc|ccc|ccc} 3 & 0 & -4 & -2 & 4 & 0 & 5 & 1 & 0 \\ 2 & 5 & -1 & -3 & 0 & 3 & 3 & 1 & 2 \\ 0 & 3 & -2 & -3 & 2 & 5 & 0 & 4 & 3 \end{array}\right)\right]$$

$$= 2\cdot\left(\begin{array}{cc|cc} 4 & 0 & 1 & 0 \\ 2 & 5 & 4 & 3 \end{array}\right) - 5\cdot\left(\begin{array}{cc|cc} -2 & 0 & 5 & 0 \\ -3 & 5 & 0 & 3 \end{array}\right) + (-1)\cdot\left(\begin{array}{cc|cc} -2 & 4 & 5 & 1 \\ -3 & 2 & 0 & 4 \end{array}\right) - (-3)\cdot\left(\begin{array}{cc|cc} 0 & -4 & 1 & 0 \\ 3 & -2 & 4 & 3 \end{array}\right)$$

$$+0\cdot\left(\begin{array}{cc|cc}3 & -4 & 5 & 0\\ 0 & -2 & 0 & 3\end{array}\right)-3\cdot\left(\begin{array}{cc|cc}3 & 0 & 5 & 1\\ 0 & 3 & 0 & 4\end{array}\right)+5\cdot\left(\begin{array}{cc|cc}5 & -1 & 0 & 3\\ 3 & -2 & 2 & 5\end{array}\right)-1\cdot\left(\begin{array}{cc|cc}2 & -1 & -3 & 3\\ 0 & -2 & -3 & 5\end{array}\right)+0\cdot\left(\begin{array}{cc|cc}2 & 5 & -3 & 0\\ 0 & 3 & -3 & 2\end{array}\right)$$

so

$$\det{(A)}=326.$$

After expanding further the minors of the above determinant based on Theorem 1, we see that this result is the same as the result of Example 2.
For $i=3$, we have:

$$\det(A)=\det\left[\left(\begin{array}{ccc|ccc|ccc}3 & 0 & -4 & -2 & 4 & 0 & 5 & 1 & 0\\ 2 & 5 & -1 & -3 & 0 & 3 & 3 & 1 & 2\\ 0 & 3 & -2 & -3 & 2 & 5 & 0 & 4 & 3\end{array}\right)\right]$$
$$=0\cdot\left(\begin{array}{cc|cc}4 & 0 & 1 & 0\\ 0 & 3 & 1 & 2\end{array}\right)-5\cdot\left(\begin{array}{cc|cc}-2 & 0 & 5 & 0\\ -3 & 3 & 3 & 2\end{array}\right)+(-2)\cdot\left(\begin{array}{cc|cc}-2 & 4 & 5 & 1\\ -3 & 0 & 3 & 1\end{array}\right)-(-3)\cdot\left(\begin{array}{cc|cc}0 & -4 & 1 & 0\\ 5 & -1 & 1 & 2\end{array}\right)$$
$$+2\cdot\left(\begin{array}{cc|cc}3 & -4 & 5 & 0\\ 2 & -1 & 3 & 2\end{array}\right)-5\cdot\left(\begin{array}{cc|cc}3 & 0 & 5 & 1\\ 2 & 5 & 3 & 1\end{array}\right)+0\cdot\left(\begin{array}{cc|cc}5 & -1 & 0 & 3\\ 5 & -1 & 0 & 3\end{array}\right)-4\cdot\left(\begin{array}{cc|cc}2 & -1 & -3 & 3\\ 2 & -1 & -3 & 3\end{array}\right)+3\cdot\left(\begin{array}{cc|cc}2 & 5 & -3 & 0\\ 2 & 5 & -3 & 0\end{array}\right)$$

So

$$\det{(A)}=326.$$

After expanding further the minors of the above determinant based on Theorem 1, we see that this result is the same as the result of Example 2.
For $j=1$, we have:

$$\det(A)=\det\left[\left(\begin{array}{ccc|ccc|ccc}3 & 0 & -4 & -2 & 4 & 0 & 5 & 1 & 0\\ 2 & 5 & -1 & -3 & 0 & 3 & 3 & 1 & 2\\ 0 & 3 & -2 & -3 & 2 & 5 & 0 & 4 & 3\end{array}\right)\right]$$
$$=3\cdot\left(\begin{array}{cc|cc}0 & 3 & 1 & 2\\ 2 & 5 & 4 & 3\end{array}\right)-2\cdot\left(\begin{array}{cc|cc}4 & 0 & 1 & 0\\ 2 & 5 & 4 & 3\end{array}\right)+0\cdot\left(\begin{array}{cc|cc}4 & 0 & 1 & 0\\ 0 & 3 & 1 & 2\end{array}\right)$$
$$-(-2)\cdot\left(\begin{array}{cc|cc}5 & -1 & 1 & 2\\ 3 & -2 & 4 & 3\end{array}\right)+(-3)\cdot\left(\begin{array}{cc|cc}0 & -4 & 1 & 0\\ 3 & -2 & 4 & 3\end{array}\right)-(-3)\cdot\left(\begin{array}{cc|cc}0 & -4 & 1 & 0\\ 5 & -1 & 1 & 2\end{array}\right)$$
$$+5\cdot\left(\begin{array}{cc|cc}5 & -1 & 0 & 3\\ 3 & -2 & 2 & 5\end{array}\right)-3\cdot\left(\begin{array}{cc|cc}0 & -4 & 4 & 0\\ 3 & -2 & 2 & 5\end{array}\right)+0\cdot\left(\begin{array}{cc|cc}0 & -4 & 4 & 0\\ 5 & -1 & 0 & 3\end{array}\right)$$

So

$$\det{(A)}=326.$$

After expanding further the minors of the above determinant based on Theorem 1, we see that this result is the same as the result of Example 2.
For $j=2$, we have:

$$\det(A)=\det\left[\left(\begin{array}{ccc|ccc|ccc}3 & 0 & -4 & -2 & 4 & 0 & 5 & 1 & 0\\ 2 & 5 & -1 & -3 & 0 & 3 & 3 & 1 & 2\\ 0 & 3 & -2 & -3 & 2 & 5 & 0 & 4 & 3\end{array}\right)\right]$$
$$=0\cdot\left(\begin{array}{cc|cc}-3 & 3 & 3 & 2\\ -3 & 5 & 0 & 3\end{array}\right)-5\cdot\left(\begin{array}{cc|cc}-2 & 0 & 5 & 0\\ -3 & 5 & 0 & 3\end{array}\right)+3\cdot\left(\begin{array}{cc|cc}-2 & 0 & 5 & 0\\ -3 & 3 & 3 & 2\end{array}\right)-4\cdot\left(\begin{array}{cc|cc}2 & -1 & 3 & 2\\ 0 & -2 & 0 & 3\end{array}\right)$$
$$+0\cdot\left(\begin{array}{cc|cc}3 & -4 & 5 & 0\\ 0 & -2 & 0 & 3\end{array}\right)-2\cdot\left(\begin{array}{cc|cc}3 & -4 & 5 & 0\\ 2 & -1 & 3 & 2\end{array}\right)+1\cdot\left(\begin{array}{cc|cc}2 & -1 & -3 & 3\\ 0 & -2 & -3 & 5\end{array}\right)-1\cdot\left(\begin{array}{cc|cc}3 & -4 & -2 & 0\\ 0 & -2 & -3 & 5\end{array}\right)$$
$$+4\cdot\left(\begin{array}{cc|cc}3 & -4 & -2 & 0\\ 2 & -1 & -3 & 3\end{array}\right)=326.$$

After expanding further the minors of the above determinant based on Theorem 1, we see that this result is the same as the result of Example 2.
For $j=3$, we have:

$$\det(A)=\det\left[\left(\begin{array}{ccc|ccc|ccc}3&0&-4&-2&4&0&5&1&0\\2&5&-1&-3&0&3&3&1&2\\0&3&-2&-3&2&5&0&4&3\end{array}\right)\right]$$
$$=(-4)\cdot\left(\begin{array}{cc|cc}-3&0&3&1\\-3&2&0&4\end{array}\right)-(-1)\cdot\left(\begin{array}{cc|cc}-2&4&5&1\\-3&2&0&4\end{array}\right)+(-2)\cdot\left(\begin{array}{cc|cc}-2&4&5&1\\-3&0&3&1\end{array}\right)-0\cdot\left(\begin{array}{cc|cc}2&5&3&1\\0&3&0&4\end{array}\right)$$
$$+3\cdot\left(\begin{array}{cc|cc}3&0&5&1\\0&3&0&4\end{array}\right)-5\cdot\left(\begin{array}{cc|cc}3&0&5&1\\2&5&3&1\end{array}\right)+0\cdot\left(\begin{array}{cc|cc}2&5&-3&0\\0&3&-3&2\end{array}\right)-2\cdot\left(\begin{array}{cc|cc}3&0&-2&4\\0&3&-3&2\end{array}\right)$$
$$+3\cdot\left(\begin{array}{cc|cc}3&0&-2&4\\2&5&-3&0\end{array}\right)=326.$$

After expanding further the minors of the above determinant based on Theorem 1, we see that this result is the same as the result of Example 2.
For $k=1$, we have:

$$\det(A)=\det\left[\left(\begin{array}{ccc|ccc|ccc}3&0&-4&-2&4&0&5&1&0\\2&5&-1&-3&0&3&3&1&2\\0&3&-2&-3&2&5&0&4&3\end{array}\right)\right]$$
$$=3\cdot\left(\begin{array}{cc|cc}0&3&1&2\\2&5&4&3\end{array}\right)-0\cdot\left(\begin{array}{cc|cc}-3&3&3&2\\-3&5&0&3\end{array}\right)+(-4)\cdot\left(\begin{array}{cc|cc}-3&0&3&1\\-3&2&0&4\end{array}\right)-2\cdot\left(\begin{array}{cc|cc}4&0&1&0\\2&5&4&3\end{array}\right)$$
$$+5\cdot\left(\begin{array}{cc|cc}-2&0&5&0\\-3&5&0&3\end{array}\right)-(-1)\cdot\left(\begin{array}{cc|cc}-2&4&5&1\\-3&2&0&4\end{array}\right)+0\cdot\left(\begin{array}{cc|cc}4&0&1&2\\0&3&4&3\end{array}\right)-3\cdot\left(\begin{array}{cc|cc}-2&0&1&0\\-3&3&3&2\end{array}\right)$$
$$+(-2)\cdot\left(\begin{array}{cc|cc}-2&4&5&1\\-3&0&3&1\end{array}\right)=326.$$

After expanding further the minors of the above determinant based on Theorem 1, we see that this result is the same as the result of Example 2.
For $k=2$, we have:

$$\det(A)=\det\left[\left(\begin{array}{ccc|ccc|ccc}3&0&-4&-2&4&0&5&1&0\\2&5&-1&-3&0&3&3&1&2\\0&3&-2&-3&2&5&0&4&3\end{array}\right)\right]$$
$$=(-2)\cdot\left(\begin{array}{cc|cc}5&-1&1&2\\3&-2&4&3\end{array}\right)-4\cdot\left(\begin{array}{cc|cc}2&-1&3&2\\0&-2&0&3\end{array}\right)+0\cdot\left(\begin{array}{cc|cc}2&5&3&1\\0&3&0&4\end{array}\right)-(-3)\cdot\left(\begin{array}{cc|cc}0&-4&1&0\\3&-2&4&3\end{array}\right)$$
$$+0\cdot\left(\begin{array}{cc|cc}3&-4&5&0\\0&-2&0&3\end{array}\right)-3\cdot\left(\begin{array}{cc|cc}3&0&5&1\\0&3&0&4\end{array}\right)+(-3)\cdot\left(\begin{array}{cc|cc}0&-4&1&2\\5&-1&4&3\end{array}\right)-2\cdot\left(\begin{array}{cc|cc}3&-4&1&0\\2&-1&3&2\end{array}\right)$$
$$+5\cdot\left(\begin{array}{cc|cc}3&0&5&1\\2&5&3&1\end{array}\right)=326\ \Rightarrow\ \mathbf{\det(A)=326.}$$

After expanding further the minors of the above determinant based on Theorem 1, we see that this result is the same as the result of Example 2.
For $k=3$, we have:

$$\det(A)=\det\left[\left(\begin{array}{ccc|ccc|ccc}3&0&-4&-2&4&0&5&1&0\\2&5&-1&-3&0&3&3&1&2\\0&3&-2&-3&2&5&0&4&3\end{array}\right)\right]$$
$$=5\cdot\left(\begin{array}{cc|cc}5&-1&0&3\\3&-2&2&5\end{array}\right)-1\cdot\left(\begin{array}{cc|cc}2&-1&-3&3\\0&-2&-3&5\end{array}\right)+0\cdot\left(\begin{array}{cc|cc}2&5&-3&0\\0&3&-3&2\end{array}\right)-3\cdot\left(\begin{array}{cc|cc}0&-4&4&0\\3&-2&2&5\end{array}\right)$$
$$+1\cdot\left(\begin{array}{cc|cc}3&-4&-2&0\\0&-2&-3&5\end{array}\right)-2\cdot\left(\begin{array}{cc|cc}3&0&-2&4\\0&3&-3&2\end{array}\right)+0\cdot\left(\begin{array}{cc|cc}0&-4&0&3\\5&-1&2&5\end{array}\right)-4\cdot\left(\begin{array}{cc|cc}3&-4&4&0\\2&-1&-3&3\end{array}\right)$$
$$+3\cdot\left(\begin{array}{cc|cc}3&0&-2&4\\2&5&-3&0\end{array}\right)\ \Rightarrow\ \mathbf{\det(A)=326.}$$

After expanding further the minors of the above determinant based on Theorem 1, we see that this result is the same as the result of Example 2.

From Theorem 1 and Theorem 2, we have true the following Theorem,

Theorem 3 *The Laplace Expansion for Determinant calculation, applies to the cubic-matrix of order 2 and the cubic matrix of order 3.*

Algorithmics Implementation of Determinants for Cubic-Matrix of order 2 and 3

In a paper (Salihu-Zaka 2023b) we have presented the pseudo-code of algorithm based on the permutation expansion method as presented in Definition 1. In the following, we have also presented the pseudo-code of the algorithm based on the Laplace method as presented in Theorem 3.

tw]

P 1: Laplace method for determinants of cubic matrices of order 2 and 3

tw]

Step 1: Determine the order of determinants:

$[m, n, o] = size(A);$

Step 2: Checking if the 3D matrix is cubic:

if $m \sim n; m \sim= o; n \sim= o;$

disp('A is not square, cannot calculate the determinant')

$d = 0;$

return

end

Step 3: Checking if the 3D matrix is higher than the 3rd order:

if $m > 3;$

disp('A is higher than the third order, hence can not be calculated.')

$d = 0;$

return

end

Step 4: Initialize $d = 0;$

Step 5: Handling base case.

if $m == 1$

$d = A;$

return

end

Step 6: Select which plan we shall use to expand the determinant:

Horizontal Layer: x1 = 1 or 2 or 3; or

Vertical Layer: x2 = 1 or 2 or 3; or

Vertical page: x3 = 1 or 2 or 3;

Step 7: Calculate the 3D determinant of orders 2 and 3 based on Laplace methodology:

Create a loop from 1 to 2 or 3 (Depending on the order of the cubic matrix):

Create a loop from 1 to 2 or 3 (Depending on the order of the cubic matrix):

If the horizontal layer is selected:

$d = d + (-1)^\wedge(1 + x1 + i + j) * A(x1, i, j) * det\_3DLaplace(A([1{:}x1 - 1x1 + 1{:}m], [1{:}i - 1i + 1{:}n], [1{:}j - 1j + 1{:}m]));$

end

If the vertical layer is selected:

$d = d + (-1)^\wedge(1 + i + x2 + j) * A(i, x2, j) * det\_3DLaplace(A([1{:}i - 1i + 1{:}m], [1{:}x2 - 1x2 + 1{:}n], [1{:}j - 1j + 1{:}m]));$

end

If the vertical page is selected:

$d = d + (-1)^\wedge(1 + i + j + x3) * A(j, i, x3) * det\_3DLaplace(A([1{:}i - 1i + 1{:}m], [1{:}j - 1j + 1{:}n], [1{:}x3 - 1x3 + 1{:}m]));$

end

end

end

Step 8: Return the result of the 3D determinant.

tw]

Conclusions

In this paper, we have continued our work on determinants of cubic-matrices. We have provided that Laplace method which is used on determinants of square and rectangular matrices similarly can be used also for the calculation of determinants of cubic-matrices of order 2 and order 3. In both cases, we have provided the proof of expanding cubic-matrix in any element, and similar to the determinant of square and rectangular matrices we have used also cubic-minors by removing the horizontal layer, vertical page and vertical layer of the corresponding element. In addition, we have also provided examples for each case, as well as we have provided a computer algorithm that can be used to calculate determinants of cubic-matrices of orders 2 and 3.

We are currently working on: systems of linear and non-linear equations with 3-dimensional representations, presenting them with 3D matrices, this way, we think, significantly reduces the representations and actions of difficult and complex problems, we are also studying '3D-matrix transformations', etc. The understanding and study of determinants for cubic-matrix, we think opens new paths for future research related to cubic-matrix applications.

We think that the concept of 3D matrices can be applied very well, in complex problems of *Game-Theory*, *Graph-Theory*, *Computer Graphics*, *Imagery*, *different problems from informatics*, *Partial differential equations*, etc. Therefore, we recommend that future research, based on this paper but also on other papers that we have for 3D-matrix, focused on the possible applications of 3D-matrices and the Determinants of cubic-matrices.

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