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The Determinant of Cubic-Matrix of Order 2 and Order 3, Some Basic Properties and Algorithms

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Abstract: Based on geometric intuition, in this paper we are trying to give an idea and visualize the meaning of the determinants for the cubic-matrix. In this paper we have analysed the possibilities of developing the concept of determinant of matrices with three indexed 3D Matrices. We define the concept of determinant for cubic-matrix of order 2 and order 3, study and prove some basic properties for calculations of determinants of cubic-matrix of order 2 and 3. Furthermore we have also tested several square determinant properties and noted that these properties also are applicable on this concept of 3D Determinants.

Keywords: 3D determinants, Determinant properties, Computer algorithm, Time complexity.

Introduction and Preliminaries

Based on the determinant of 2D square matrices (Salihu et al., 2021b; Artin, 1991; Bretscher, 2005; Salihu, 2018; Schneide et al., 1973; Salihu et al., 2019; Lang, 1987). As well as determinant of rectangular matrices (Radić, 1966; Salihu et al., 2022b, Salihu et al., 2023; Radić, 2005; Salihu et al., 2022a; Salihu et al., 2022c; Salihu et al., 2021a; Amiri et al., 2010; Makarewicz et al., 2014). We have come to the idea of developing the concept of determinant of 3D cubic matrices, our concept is based on permutation expansion method. Encouraged by geometric intuition, in this paper we are trying to give an idea and visualize the meaning of the determinants for the cubic-matrix. Our early research mainly lies between geometry, algebra, matrix theory, etc., (see Peters-Zaka, 2023; Zaka, 2019a; Zaka-Filipi, 2016; Filipi et al., 2019; Zaka, 2017b; Zaka, 2018; Zaka, 2016; Zaka-Peters, 2020; Zaka-Peters, 2021; Zaka et al., 2020a; Zaka et al., 2020b; Zaka-Peters, 2024a; Zaka-Peters, 2024b; Deda-Zaka, 2024). This paper is continuation of the ideas that arise based on previous researches of 3D matrix ring with element from any whatever field F see (Zaka, 2017a). But here we study the case when the field F is the field of real numbers R. In the paper (Salihu-Zaka, 2023; Zaka-Salihu, 2024).

We have made progress in our research, related to cubic-matrix determinants, we have studied some basic properties related to determinants of cubic matrix, we have also tried how the Laplace expansion method works in the calculation of cubic-matrix determinants for cubic-matrix of order 2 and 3. In this paper we follow a different method from the calculation of determinants of 3D matrix, which is studied in (Zaka, 2019b). In contrast to the meaning of the determinant as a multi-scalar studied in (Zaka, 2019b). İn this paper we give a new definition, for the determinant of the 3D-cubic-matrix, which is a real-number. In the papers (Zaka, 2017a; Zaka, 2019b). Have been studied in detail, properties for 3D-matrix, therefore, those studied properties are also valid for 3D-cubic-Matrix. Our point in this paper is to provide a concept of determinant of 3D matrices. Our concept is based on Milne-Thomson (see Milne-Thomson, 1941). Or permutation method used in regular square matrices.

The following is definition of 3D matrices provided by Zaka in 2017, see (Zaka, 2017a; Zaka, 2019b).

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Definition 1 *3-dimensional $m \times n \times p$ matrix will call, a matrix which has: m-horizontal layers (analogous to m-rows), n-vertical page (analogue with n- columns in the usual matrices) and p-vertical layers (p-1 of which are hidden).*

The set of these matrix's the write how:

$$M_{m \times n \times p}(\mathrm{F}) = \{a_{i,j,k} | a_{i,j,k} \in F - field\ \forall i = \overline{1, m}; \quad j = \overline{1, n}; \quad k = \overline{1, p}\}. \tag{1}$$

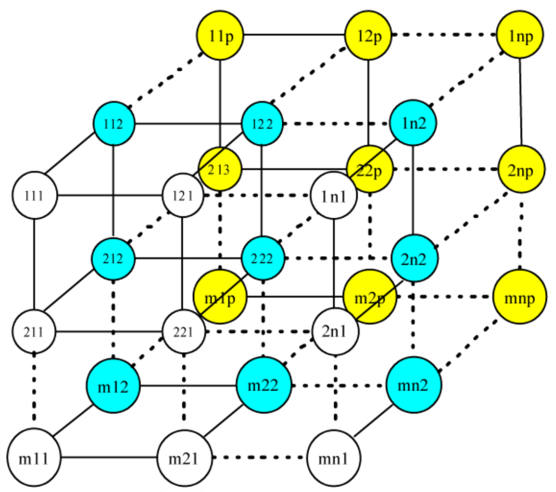

Figure 1. 3D-*Matrix view.*

In the following is presented the determinant of 3D-cubic matrices, as well several properties which are adopted from 2D square determinants.

Cubic-Matrix of Order 2 and 3 and Their Determinants

A 3-dimensional-matrix $A_{n \times n \times n}$ for $n = 2,3, \ldots$, called "cubic-matrix of order n". For $n = 1$ we have that the cubic-matrix of order 1 is an element of F.

Let us now consider the set of cubic-matrix of order n, for $n = 2$ or $n = 3$, with elements from a field F (so when cubic-matrix of order n, there are: n −vertical pages, n −horizontal layers and n −vertical layers). From (Zaka, 2017a; Zaka, 2019b). We have that, the addition of 3D-matrix stands also for cubic-matrix of order 2 and 3. Also, the set of cubic-matrix of order 2 and 3 forms an commutative group (Abelian Group) related to 3Dmatrix addition.

Determinants of Cubic-Matrix of Order 2 and 3

In paper (Salihu-Zaka, 2023). We have defined and described the meaning of the determinants of cubic-matrix of order 2 and order 3, with elements from a field F. Recall that a cubic-matrix $A_{n \times n \times n}$ for $n = 2,3, \ldots$, called "cubic-matrix of order n".

For $n = 1$ we have that the cubic-matrix of order 1 is an element of F.

Let us now consider the set of cubic-matrix of order n, with elements from a field F (so when cubic-matrix of order n, there are: n −vertical pages, n −horizontal layers and n −vertical layers),

$$\mathcal{M}_n(F) = \{A_{n \times n \times n} = (a_{ijk})_{n \times n \times n} | a_{ijk} \in F, \ \forall i = \overline{1, n}; j = \overline{1, n}; k = \overline{1, n}\}.$$

In this paper, we define the *determinant of cubic-matrix* as a element from this field, so the map,

$$\begin{aligned} \det: \mathcal{M}_n(F) &\to F \\ \forall A \in \mathcal{M}_n(F) &\mapsto \det(A) \in F. \end{aligned}$$

Below we give two definitions, how we will calculate the determinant of the cubic-matrix of orders 2 and 3.

Definition 2 *Let $A \in \mathcal{M}_2(F)$ be a $2 \times 2 \times 2$, with elements from a field F.*

$$A_{2\times2\times2} = \left(\begin{matrix} a_{111} & a_{121} \\ a_{211} & a_{221} \end{matrix}\middle| \begin{matrix} a_{112} & a_{122} \\ a_{212} & a_{222} \end{matrix}\right).$$

Determinant of this cubic-matrix, we called,

$$\det[A_{2\times2\times2}] = \det\left(\begin{matrix} a_{111} & a_{121} \\ a_{211} & a_{221} \end{matrix}\middle| \begin{matrix} a_{112} & a_{122} \\ a_{212} & a_{222} \end{matrix}\right) = a_{111}\cdot a_{222} - a_{112}\cdot a_{221} - a_{121}\cdot a_{212} + a_{122}\cdot a_{211}.$$

The follow example is case where cubic-matrix, is with elements from the number field $\mathbb{R}$.

Example 1 *Let's have the cubic-matrix, with element in number field $\mathbb{R}$,*

$$\det[A_{2\times2\times2}] = \det\left(\begin{matrix} 4 & -3 \\ -1 & 5 \end{matrix}\middle| \begin{matrix} -2 & 4 \\ -7 & 3 \end{matrix}\right).$$

then according to the definition 2, we calculate the Determinant of this cubic-matrix, and have,

$$\det[A_{2\times2\times2}] = \det\left(\begin{matrix} 4 & -3 \\ -1 & 5 \end{matrix}\middle| \begin{matrix} -2 & 4 \\ -7 & 3 \end{matrix}\right) = 4\cdot3 - (-2)\cdot5 - (-3)\cdot(-7) + 4\cdot(-1)$$

$$\det[A_{2\times2\times2}] = 12 - (-10) - 21 + (-4) = 12 + 10 - 21 - 4 = -3.$$

We are trying to expand the meaning of the determinant of cubic-matrix, for order 3 (so when cubic-matrix, there are: 3-vertical pages, 3-horizontal layers and 3-vertical layers).

Definition 3 *Let $A \in \mathcal{M}_3(F)$ be a $3 \times 3 \times 3$ cubic-matrix with element from a field F,*

$$A_{3\times3\times3} = \left(\begin{matrix} a_{111} & a_{121} & a_{131} \\ a_{211} & a_{221} & a_{231} \\ a_{311} & a_{321} & a_{331} \end{matrix}\middle| \begin{matrix} a_{112} & a_{122} & a_{132} \\ a_{212} & a_{222} & a_{232} \\ a_{312} & a_{322} & a_{332} \end{matrix}\middle| \begin{matrix} a_{113} & a_{123} & a_{133} \\ a_{213} & a_{223} & a_{233} \\ a_{313} & a_{323} & a_{333} \end{matrix}\right).$$

Determinant of this cubic-matrix, we called,

$$\det[A_{3\times3\times3}] = \det\left(\begin{matrix} a_{111} & a_{121} & a_{131} \\ a_{211} & a_{221} & a_{231} \\ a_{311} & a_{321} & a_{331} \end{matrix}\middle| \begin{matrix} a_{112} & a_{122} & a_{132} \\ a_{212} & a_{222} & a_{232} \\ a_{312} & a_{322} & a_{332} \end{matrix}\middle| \begin{matrix} a_{113} & a_{123} & a_{133} \\ a_{213} & a_{223} & a_{233} \\ a_{313} & a_{323} & a_{333} \end{matrix}\right). \quad (2)$$

$$\det[A_{3\times3\times3}] = a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332}$$

$$+a_{111}\cdot a_{233}\cdot a_{322} - a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331}$$

$$+a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321} + a_{113}\cdot a_{221}\cdot a_{332}$$

$$-a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321}$$

$$-a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313}$$

$$-a_{121}\cdot a_{233}\cdot a_{312} + a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331}$$

$$-a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311} - a_{123}\cdot a_{211}\cdot a_{332}$$

$$+a_{123}\cdot a_{212}\cdot a_{331}+a_{123}\cdot a_{231}\cdot a_{312}-a_{123}\cdot a_{232}\cdot a_{311}$$

$$+a_{131}\cdot a_{212}\cdot a_{323}-a_{131}\cdot a_{213}\cdot a_{322}-a_{131}\cdot a_{222}\cdot a_{313}$$

$$+a_{131}\cdot a_{223}\cdot a_{312}-a_{132}\cdot a_{211}\cdot a_{323}+a_{132}\cdot a_{213}\cdot a_{321}$$

$$+a_{132}\cdot a_{221}\cdot a_{313}-a_{132}\cdot a_{223}\cdot a_{311}+a_{133}\cdot a_{211}\cdot a_{322}$$

$$-a_{133}\cdot a_{212}\cdot a_{321}-a_{133}\cdot a_{221}\cdot a_{312}+a_{133}\cdot a_{222}\cdot a_{311}.$$

The follow example is case where cubic-matrix, is with elements from the number field $\mathbb{R}$.

Example 2 *Let's have the cubic-matrix of order 3, with element from number field (field of real numbers)* $\mathbb{R}$,

$$\det[A_{3\times3\times3}]=\det\left(\begin{matrix}3&0&-4\\2&5&-1\\0&3&-2\end{matrix}\middle|\begin{matrix}-2&4&0\\-3&0&3\\-3&2&5\end{matrix}\middle|\begin{matrix}5&1&0\\3&1&2\\0&4&3\end{matrix}\right).$$

Then, we calculate the Determinant of this cubic-matrix following the Definition 3, and have that,

$$\det[A_{3\times3\times3}]=\det\left(\begin{matrix}3&0&-4\\2&5&-1\\0&3&-2\end{matrix}\middle|\begin{matrix}-2&4&0\\-3&0&3\\-3&2&5\end{matrix}\middle|\begin{matrix}5&1&0\\3&1&2\\0&4&3\end{matrix}\right)$$

$$\begin{gathered}=3\cdot0\cdot3-3\cdot3\cdot4-3\cdot1\cdot5+3\cdot2\cdot2-(-2)5\cdot3+(-2)1(-2)+(-2)(-1)\cdot4-(-2)2\cdot3\\+5\cdot5\cdot5-5\cdot0\cdot(-2)-5\cdot(-1)\cdot2+5\cdot3\cdot3-0\cdot(-3)\cdot3+0\cdot3\cdot5+0\cdot3\cdot0-0\cdot2\cdot(-3)\\+4\cdot2\cdot3-4\cdot3\cdot(-2)-4\cdot(-1)\cdot0+4\cdot2\cdot0-1\cdot2\cdot5+1\cdot(-3)\cdot(-2)+1\cdot(-1)\cdot(-3)-1\cdot3\cdot0\\+(-4)(-3)\cdot4-(-4)3\cdot2-(-4)0\cdot0+(-4)1(-3)-0\cdot2\cdot4+0\cdot3\cdot3+0\cdot5\cdot0-0\cdot1\cdot0+\end{gathered}$$

$$+0\cdot2\cdot2-0\cdot(-3)\cdot3-0\cdot5\cdot(-3)+0\cdot0\cdot0$$

so,

$$\begin{aligned}\det[A_{3\times3\times3}]=&\ 0-36-15+12+30+4+8+12+125+0+10+45+0+0+0+0+24+24+0\\&+0-10+6+3-0+48+24+0+12-0+0+0-0+0+0+0+0=326.\end{aligned}$$

Hence,

$$\det\left(\begin{matrix}3&0&-4\\2&5&-1\\0&3&-2\end{matrix}\middle|\begin{matrix}-2&4&0\\-3&0&3\\-3&2&5\end{matrix}\middle|\begin{matrix}5&1&0\\3&1&2\\0&4&3\end{matrix}\right)=326.$$

Some Basic Properties of Determinants for Cubic-Matrix of Order 2 and Order 3

Definition 4 *We will call* I_n, *a unit-3D-cubic-matrix of order 2 or 3, with elements* e_{ijk}, *which are:*

$$e_{ijk}=\begin{cases}0 & for,\quad i\neq j\neq k\\1 & for,\quad i=j=k\end{cases}$$

Proposition 1 *For every unit-cubic-matrix of order 2 or 3, with element from number field* $\mathbb{R}$, *we have that* $det(I_n)=1$.

Let's have the unit cubic-matrix of order 2,

$$I_2=\left(\begin{matrix}1&0\\0&0\end{matrix}\middle|\begin{matrix}0&0\\0&1\end{matrix}\right)$$

then, this determinant is,

$$\det(I_2) = \det\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = 1 \cdot 1 = 1.$$

Now lets have the unit-cubic matrix of order 3,

$$I_3 = \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

then this determinant is,

$$\det(I_3) = \det[I_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) = 1 \cdot 1 \cdot 1 = 1$$

Theorem 1 *Suppose that A is a cubic-matrix of order 2 or 3, with a plan where every entry is zero then its determinant is 'zero', so $\det(A) = 0$.*

Proof. We are discussing the following cases:

1. For plan $i = 1$: Let A be cubic-matrix of order 2, where all elements on the plan $i = 1$ are equal to zero, then based on definition 2:

$$\det[A_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right) = 0 \cdot a_{222} - 0 \cdot a_{221} - 0 \cdot a_{212} + 0 \cdot a_{211} = 0.$$

2. For plan $i = 2$: Let A be cubic-matrix of order 2, where all elements on the plan $i = 2$ are equal to zero, then based on definition 2:

$$A_{2\times2\times2} = \left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ 0 & 0 & 0 & 0 \end{array}\right) = a_{111} \cdot 0 - a_{112} \cdot 0 - a_{121} \cdot 0 + a_{122} \cdot 0 = 0.$$

3. For plan $j = 1$: Let A be cubic-matrix of order 2, where all elements on the plan $j = 1$ are equal to zero, then based on definition 2:

$$\det[A_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} 0 & a_{121} & 0 & a_{122} \\ 0 & a_{221} & 0 & a_{222} \end{array}\right) = 0 \cdot a_{222} - 0 \cdot a_{221} - a_{121} \cdot 0 + a_{122} \cdot 0 = 0.$$

4. For plan $j = 2$: Let A be cubic-matrix of order 2, where all elements on the plan $j = 2$ are equal to zero, then based on definition 2:

$$\det[A_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111} & 0 & a_{112} & 0 \\ a_{211} & 0 & a_{212} & 0 \end{array}\right) = a_{111} \cdot 0 - a_{112} \cdot 0 - 0 \cdot a_{212} + 0 \cdot a_{211} = 0.$$

5. For plan $k = 1$: Let A be cubic-matrix of order 2, where all elements on the plan $k = 1$ are equal to zero, then based on definition 2:

$$\det[A_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} 0 & 0 & a_{112} & a_{122} \\ 0 & 0 & a_{212} & a_{222} \end{array}\right) = 0 \cdot a_{222} - a_{112} \cdot 0 - 0 \cdot a_{212} + a_{122} \cdot 0 = 0.$$

6. For plan $k = 2$: Let A be cubic-matrix of order 2, where all elements on the plan $k = 2$ are equal to zero, then based on definition 2:

$$\det[A_{2\times2\times2}] = \det\left(\begin{array}{cc|cc} a_{111} & a_{121} & 0 & 0 \\ a_{211} & a_{221} & 0 & 0 \end{array}\right) = a_{111} \cdot 0 - 0 \cdot a_{221} - a_{121} \cdot 0 + 0 \cdot a_{211} = 0.$$

Now we will consider for ***third order*** cubic-matrix, as following.

1. For plan $i = 1$: Let A be cubic-matrix of order 3, where all elements on the plan $i = 1$ are equal to zero, then based on definition 3:

$$\det[A_{3\times3\times3}] = \det\left(\begin{matrix} 0 & 0 & 0 \\ a_{211} & a_{221} & a_{231} \\ a_{311} & a_{321} & a_{331} \end{matrix}\middle| \begin{matrix} 0 & 0 & 0 \\ a_{212} & a_{222} & a_{232} \\ a_{312} & a_{322} & a_{332} \end{matrix}\middle| \begin{matrix} 0 & 0 & 0 \\ a_{213} & a_{223} & a_{233} \\ a_{313} & a_{323} & a_{333} \end{matrix}\right)$$

$$= 0 \cdot a_{222} \cdot a_{333} - 0 \cdot a_{232} \cdot a_{323} - 0 \cdot a_{223} \cdot a_{332} + 0 \cdot a_{233} \cdot a_{322}$$

$$-0 \cdot a_{221} \cdot a_{333} + 0 \cdot a_{223} \cdot a_{331} + 0 \cdot a_{231} \cdot a_{323} - 0 \cdot a_{233} \cdot a_{321}$$

$$+0 \cdot a_{221} \cdot a_{332} - 0 \cdot a_{222} \cdot a_{331} - 0 \cdot a_{231} \cdot a_{322} + 0 \cdot a_{232} \cdot a_{321}$$

$$-0 \cdot a_{212} \cdot a_{333} + 0 \cdot a_{213} \cdot a_{332} + 0 \cdot a_{232} \cdot a_{313} - 0 \cdot a_{233} \cdot a_{312}$$

$$+0 \cdot a_{211} \cdot a_{333} - 0 \cdot a_{213} \cdot a_{331} - 0 \cdot a_{231} \cdot a_{313} + 0 \cdot a_{233} \cdot a_{311}$$

$$-0 \cdot a_{211} \cdot a_{332} + 0 \cdot a_{212} \cdot a_{331} + 0 \cdot a_{231} \cdot a_{312} - 0 \cdot a_{232} \cdot a_{311}$$

$$+0 \cdot a_{212} \cdot a_{323} - 0 \cdot a_{213} \cdot a_{322} - 0 \cdot a_{222} \cdot a_{313} + 0 \cdot a_{223} \cdot a_{312}$$

$$-0 \cdot a_{211} \cdot a_{323} + 0 \cdot a_{213} \cdot a_{321} + 0 \cdot a_{221} \cdot a_{313} - 0 \cdot a_{223} \cdot a_{311}$$

$$+0 \cdot a_{211} \cdot a_{322} - 0 \cdot a_{212} \cdot a_{321} - 0 \cdot a_{221} \cdot a_{312} + 0 \cdot a_{222} \cdot a_{311} = 0.$$

2. For plan $i = 2$: Let A be cubic-matrix of order 3, where all elements on the plan $i = 2$ are equal to zero, then based on definition 3:

$$\det[A_{3\times3\times3}] = \det\left(\begin{matrix} a_{111} & a_{121} & a_{131} \\ 0 & 0 & 0 \\ a_{311} & a_{321} & a_{331} \end{matrix}\middle| \begin{matrix} a_{112} & a_{122} & a_{132} \\ 0 & 0 & 0 \\ a_{312} & a_{322} & a_{332} \end{matrix}\middle| \begin{matrix} a_{113} & a_{123} & a_{133} \\ 0 & 0 & 0 \\ a_{313} & a_{323} & a_{333} \end{matrix}\right)$$

$$= a_{111} \cdot 0 \cdot a_{333} - a_{111} \cdot 0 \cdot a_{323} - a_{111} \cdot 0 \cdot a_{332} + a_{111} \cdot 0 \cdot a_{322}$$

$$-a_{112} \cdot 0 \cdot a_{333} + a_{112} \cdot 0 \cdot a_{331} + a_{112} \cdot 0 \cdot a_{323} - a_{112} \cdot 0 \cdot a_{321}$$

$$+a_{113} \cdot 0 \cdot a_{332} - a_{113} \cdot 0 \cdot a_{331} - a_{113} \cdot 0 \cdot a_{322} + a_{113} \cdot 0 \cdot a_{321}$$

$$-a_{121} \cdot 0 \cdot a_{333} + a_{121} \cdot 0 \cdot a_{332} + a_{121} \cdot 0 \cdot a_{313} - a_{121} \cdot 0 \cdot a_{312}$$

$$+a_{122} \cdot 0 \cdot a_{333} - a_{122} \cdot 0 \cdot a_{331} - a_{122} \cdot 0 \cdot a_{313} + a_{122} \cdot 0 \cdot a_{311}$$

$$-a_{123} \cdot 0 \cdot a_{332} + a_{123} \cdot 0 \cdot a_{331} + a_{123} \cdot 0 \cdot a_{312} - a_{123} \cdot 0 \cdot a_{311}$$

$$+a_{131} \cdot 0 \cdot a_{323} - a_{131} \cdot 0 \cdot a_{322} - a_{131} \cdot 0 \cdot a_{313} + a_{131} \cdot 0 \cdot a_{312}$$

$$-a_{132} \cdot 0 \cdot a_{323} + a_{132} \cdot 0 \cdot a_{321} + a_{132} \cdot 0 \cdot a_{313} - a_{132} \cdot 0 \cdot a_{311}$$

$$+a_{133} \cdot 0 \cdot a_{322} - a_{133} \cdot 0 \cdot a_{321} - a_{133} \cdot 0 \cdot a_{312} + a_{133} \cdot 0 \cdot a_{311} = 0.$$

3. For plan $i = 3$: Let A be cubic-matrix of order 3, where all elements on the plan $i = 3$ are equal to zero, then based on definition 3:

$$\det[A_{3\times3\times3}] = \det\left(\begin{matrix} a_{111} & a_{121} & a_{131} \\ a_{211} & a_{221} & a_{231} \\ 0 & 0 & 0 \end{matrix}\middle| \begin{matrix} a_{112} & a_{122} & a_{132} \\ a_{212} & a_{222} & a_{232} \\ 0 & 0 & 0 \end{matrix}\middle| \begin{matrix} a_{113} & a_{123} & a_{133} \\ a_{213} & a_{223} & a_{233} \\ 0 & 0 & 0 \end{matrix}\right)$$

$$= a_{111} \cdot a_{222} \cdot 0 - a_{111} \cdot a_{232} \cdot 0 - a_{111} \cdot a_{223} \cdot 0 + a_{111} \cdot a_{233} \cdot 0$$

$$-a_{112} \cdot a_{221} \cdot 0 + a_{112} \cdot a_{223} \cdot 0 + a_{112} \cdot a_{231} \cdot 0 - a_{112} \cdot a_{233} \cdot 0$$

$$+a_{113} \cdot a_{221} \cdot 0 - a_{113} \cdot a_{222} \cdot 0 - a_{113} \cdot a_{231} \cdot 0 + a_{113} \cdot a_{232} \cdot 0$$

$$-a_{121} \cdot a_{212} \cdot 0 + a_{121} \cdot a_{213} \cdot 0 + a_{121} \cdot a_{232} \cdot 0 - a_{121} \cdot a_{233} \cdot 0$$

$$+a_{122} \cdot a_{211} \cdot 0 - a_{122} \cdot a_{213} \cdot 0 - a_{122} \cdot a_{231} \cdot 0 + a_{122} \cdot a_{233} \cdot 0$$

$$-a_{123} \cdot a_{211} \cdot 0 + a_{123} \cdot a_{212} \cdot 0 + a_{123} \cdot a_{231} \cdot 0 - a_{123} \cdot a_{232} \cdot 0$$

$$+a_{131} \cdot a_{212} \cdot 0 - a_{131} \cdot a_{213} \cdot 0 - a_{131} \cdot a_{222} \cdot 0 + a_{131} \cdot a_{223} \cdot 0$$

$$-a_{132} \cdot a_{211} \cdot 0 + a_{132} \cdot a_{213} \cdot 0 + a_{132} \cdot a_{221} \cdot 0 - a_{132} \cdot a_{223} \cdot 0$$

$$+a_{133} \cdot a_{211} \cdot 0 - a_{133} \cdot a_{212} \cdot 0 - a_{133} \cdot a_{221} \cdot 0 + a_{133} \cdot a_{222} \cdot 0 = 0.$$

4. For plan $j = 1$: Let A be cubic-matrix of order 3, where all elements on the plan $j = 1$ are equal to zero, then based on definition 3:

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} 0 & a_{121} & a_{131} & 0 & a_{122} & a_{132} & 0 & a_{123} & a_{133} \\ 0 & a_{221} & a_{231} & 0 & a_{222} & a_{232} & 0 & a_{223} & a_{233} \\ 0 & a_{321} & a_{331} & 0 & a_{322} & a_{332} & 0 & a_{323} & a_{333} \end{array}\right)$$

$$= 0 \cdot a_{222} \cdot a_{333} - 0 \cdot a_{232} \cdot a_{323} - 0 \cdot a_{223} \cdot a_{332} + 0 \cdot a_{233} \cdot a_{322}$$

$$-0 \cdot a_{221} \cdot a_{333} + 0 \cdot a_{223} \cdot a_{331} + 0 \cdot a_{231} \cdot a_{323} - 0 \cdot a_{233} \cdot a_{321}$$

$$+0 \cdot a_{221} \cdot a_{332} - 0 \cdot a_{222} \cdot a_{331} - 0 \cdot a_{231} \cdot a_{322} + 0 \cdot a_{232} \cdot a_{321}$$

$$-a_{121} \cdot 0 \cdot a_{333} + a_{121} \cdot 0 \cdot a_{332} + a_{121} \cdot a_{232} \cdot 0 - a_{121} \cdot a_{233} \cdot 0$$

$$+a_{122} \cdot 0 \cdot a_{333} - a_{122} \cdot 0 \cdot a_{331} - a_{122} \cdot a_{231} \cdot 0 + a_{122} \cdot a_{233} \cdot 0$$

$$-a_{123} \cdot 0 \cdot a_{332} + a_{123} \cdot 0 \cdot a_{331} + a_{123} \cdot a_{231} \cdot 0 - a_{123} \cdot a_{232} \cdot 0$$

$$+a_{131} \cdot 0 \cdot a_{323} - a_{131} \cdot 0 \cdot a_{322} - a_{131} \cdot a_{222} \cdot 0 + a_{131} \cdot a_{223} \cdot 0$$

$$-a_{132} \cdot 0 \cdot a_{323} + a_{132} \cdot 0 \cdot a_{321} + a_{132} \cdot a_{221} \cdot 0 - a_{132} \cdot a_{223} \cdot 0$$

$$+a_{133} \cdot 0 \cdot a_{322} - a_{133} \cdot 0 \cdot a_{321} - a_{133} \cdot a_{221} \cdot 0 + a_{133} \cdot a_{222} \cdot 0 = 0.$$

5. For plan $j = 2$: Let A be cubic-matrix of order 3, where all elements on the plan $j = 2$ are equal to zero, then based on definition 3:

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & 0 & a_{131} & a_{112} & 0 & a_{132} & a_{113} & 0 & a_{133} \\ a_{211} & 0 & a_{231} & a_{212} & 0 & a_{232} & a_{213} & 0 & a_{233} \\ a_{311} & 0 & a_{331} & a_{312} & 0 & a_{332} & a_{313} & 0 & a_{333} \end{array}\right)$$

$$= a_{111} \cdot 0 \cdot a_{333} - a_{111} \cdot a_{232} \cdot 0 - a_{111} \cdot 0 \cdot a_{332} + a_{111} \cdot a_{233} \cdot 0$$

$$-a_{112} \cdot 0 \cdot a_{333} + a_{112} \cdot 0 \cdot a_{331} + a_{112} \cdot a_{231} \cdot 0 - a_{112} \cdot a_{233} \cdot 0$$

$$+a_{113} \cdot 0 \cdot a_{332} - a_{113} \cdot 0 \cdot a_{331} - a_{113} \cdot a_{231} \cdot 0 + a_{113} \cdot a_{232} \cdot 0$$

$$-0\cdot a_{212}\cdot a_{333}+0\cdot a_{213}\cdot a_{332}+0\cdot a_{232}\cdot a_{313}-0\cdot a_{233}\cdot a_{312}$$

$$+0\cdot a_{211}\cdot a_{333}-0\cdot a_{213}\cdot a_{331}-0\cdot a_{231}\cdot a_{313}+0\cdot a_{233}\cdot a_{311}$$

$$-0\cdot a_{211}\cdot a_{332}+0\cdot a_{212}\cdot a_{331}+0\cdot a_{231}\cdot a_{312}-0\cdot a_{232}\cdot a_{311}$$

$$+a_{131}\cdot a_{212}\cdot 0-a_{131}\cdot a_{213}\cdot 0-a_{131}\cdot 0\cdot a_{313}+a_{131}\cdot 0\cdot a_{312}$$

$$-a_{132}\cdot a_{211}\cdot 0+a_{132}\cdot a_{213}\cdot 0+a_{132}\cdot 0\cdot a_{313}-a_{132}\cdot 0\cdot a_{311}$$

$$+a_{133}\cdot a_{211}\cdot 0-a_{133}\cdot a_{212}\cdot 0-a_{133}\cdot 0\cdot a_{312}+a_{133}\cdot 0\cdot a_{311}=0.$$

6. For plan $j=3$: Let A be cubic-matrix of order 3, where all elements on the plan $j=3$ are equal to zero, then based on definition 3:

$$\det[A_{3\times3\times3}]=\det\left(\begin{matrix}a_{111}&a_{121}&0\\a_{211}&a_{221}&0\\a_{311}&a_{321}&0\end{matrix}\middle|\begin{matrix}a_{112}&a_{122}&0\\a_{212}&a_{222}&0\\a_{312}&a_{322}&0\end{matrix}\middle|\begin{matrix}a_{113}&a_{123}&0\\a_{213}&a_{223}&0\\a_{313}&a_{323}&0\end{matrix}\right)$$

$$=a_{111}\cdot a_{222}\cdot 0-a_{111}\cdot 0\cdot a_{323}-a_{111}\cdot a_{223}\cdot 0+a_{111}\cdot 0\cdot a_{322}$$

$$-a_{112}\cdot a_{221}\cdot 0+a_{112}\cdot a_{223}\cdot 0+a_{112}\cdot 0\cdot a_{323}-a_{112}\cdot 0\cdot a_{321}$$

$$+a_{113}\cdot a_{221}\cdot 0-a_{113}\cdot a_{222}\cdot 0-a_{113}\cdot 0\cdot a_{322}+a_{113}\cdot 0\cdot a_{321}$$

$$-a_{121}\cdot a_{212}\cdot 0+a_{121}\cdot a_{213}\cdot 0+a_{121}\cdot 0\cdot a_{313}-a_{121}\cdot 0\cdot a_{312}$$

$$+a_{122}\cdot a_{211}\cdot 0-a_{122}\cdot a_{213}\cdot 0-a_{122}\cdot 0\cdot a_{313}+a_{122}\cdot 0\cdot a_{311}$$

$$-a_{123}\cdot a_{211}\cdot 0+a_{123}\cdot a_{212}\cdot 0+a_{123}\cdot 0\cdot a_{312}-a_{123}\cdot 0\cdot a_{311}$$

$$+0\cdot a_{212}\cdot a_{323}-0\cdot a_{213}\cdot a_{322}-0\cdot a_{222}\cdot a_{313}+0\cdot a_{223}\cdot a_{312}$$

$$-0\cdot a_{211}\cdot a_{323}+0\cdot a_{213}\cdot a_{321}+0\cdot a_{221}\cdot a_{313}-0\cdot a_{223}\cdot a_{311}$$

$$+0\cdot a_{211}\cdot a_{322}-0\cdot a_{212}\cdot a_{321}-0\cdot a_{221}\cdot a_{312}+0\cdot a_{222}\cdot a_{311}=0.$$

7. For plan $k=1$: Let A be cubic-matrix of order 3, where all elements on the plan $k=1$ are equal to zero, then based on definition 3:

$$\det[A_{3\times3\times3}]=\det\left(\begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix}\middle|\begin{matrix}a_{112}&a_{122}&a_{132}\\a_{212}&a_{222}&a_{232}\\a_{312}&a_{322}&a_{332}\end{matrix}\middle|\begin{matrix}a_{113}&a_{123}&a_{133}\\a_{213}&a_{223}&a_{233}\\a_{313}&a_{323}&a_{333}\end{matrix}\right)$$

$$=0\cdot a_{222}\cdot a_{333}-0\cdot a_{232}\cdot a_{323}-0\cdot a_{223}\cdot a_{332}+0\cdot a_{233}\cdot a_{322}$$

$$-a_{112}\cdot 0\cdot a_{333}+a_{112}\cdot a_{223}\cdot 0+a_{112}\cdot 0\cdot a_{323}-a_{112}\cdot a_{233}\cdot 0$$

$$+a_{113}\cdot 0\cdot a_{332}-a_{113}\cdot a_{222}\cdot 0-a_{113}\cdot 0\cdot a_{322}+a_{113}\cdot a_{232}\cdot 0$$

$$-0\cdot a_{212}\cdot a_{333}+0\cdot a_{213}\cdot a_{332}+0\cdot a_{232}\cdot a_{313}-0\cdot a_{233}\cdot a_{312}$$

$$+a_{122}\cdot 0\cdot a_{333}-a_{122}\cdot a_{213}\cdot 0-a_{122}\cdot 0\cdot a_{313}+a_{122}\cdot a_{233}\cdot 0$$

$$-a_{123}\cdot 0\cdot a_{332}+a_{123}\cdot a_{212}\cdot 0+a_{123}\cdot 0\cdot a_{312}-a_{123}\cdot a_{232}\cdot 0$$

$$+0 \cdot a_{212} \cdot a_{323} - 0 \cdot a_{213} \cdot a_{322} - 0 \cdot a_{222} \cdot a_{313} + 0 \cdot a_{223} \cdot a_{312}$$

$$-a_{132} \cdot 0 \cdot a_{323} + a_{132} \cdot a_{213} \cdot 0 + a_{132} \cdot 0 \cdot a_{313} - a_{132} \cdot a_{223} \cdot 0$$

$$+a_{133} \cdot 0 \cdot a_{322} - a_{133} \cdot a_{212} \cdot 0 - a_{133} \cdot 0 \cdot a_{312} + a_{133} \cdot a_{222} \cdot 0 = 0.$$

8. For plan $k = 2$: Let A be cubic-matrix of order 3, where all elements on the plan $k = 2$ are equal to zero, then based on definition 3:

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & 0 & 0 & 0 & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & 0 & 0 & 0 & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & 0 & 0 & 0 & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$= a_{111} \cdot 0 \cdot a_{333} - a_{111} \cdot 0 \cdot a_{323} - a_{111} \cdot a_{223} \cdot 0 + a_{111} \cdot a_{233} \cdot 0$$

$$-0 \cdot a_{221} \cdot a_{333} + 0 \cdot a_{223} \cdot a_{331} + 0 \cdot a_{231} \cdot a_{323} - 0 \cdot a_{233} \cdot a_{321}$$

$$+a_{113} \cdot a_{221} \cdot 0 - a_{113} \cdot 0 \cdot a_{331} - a_{113} \cdot a_{231} \cdot 0 + a_{113} \cdot 0 \cdot a_{321}$$

$$-a_{121} \cdot 0 \cdot a_{333} + a_{121} \cdot a_{213} \cdot 0 + a_{121} \cdot 0 \cdot a_{313} - a_{121} \cdot a_{233} \cdot 0$$

$$+0 \cdot a_{211} \cdot a_{333} - 0 \cdot a_{213} \cdot a_{331} - 0 \cdot a_{231} \cdot a_{313} + 0 \cdot a_{233} \cdot a_{311}$$

$$-a_{123} \cdot a_{211} \cdot 0 + a_{123} \cdot 0 \cdot a_{331} + a_{123} \cdot a_{231} \cdot 0 - a_{123} \cdot 0 \cdot a_{311}$$

$$+a_{131} \cdot 0 \cdot a_{323} - a_{131} \cdot a_{213} \cdot 0 - a_{131} \cdot 0 \cdot a_{313} + a_{131} \cdot a_{223} \cdot 0$$

$$-0 \cdot a_{211} \cdot a_{323} + 0 \cdot a_{213} \cdot a_{321} + 0 \cdot a_{221} \cdot a_{313} - 0 \cdot a_{223} \cdot a_{311}$$

$$+a_{133} \cdot a_{211} \cdot 0 - a_{133} \cdot 0 \cdot a_{321} - a_{133} \cdot a_{221} \cdot 0 + a_{133} \cdot 0 \cdot a_{311} = 0.$$

9. For plan $k = 3$: Let A be cubic-matrix of order 3, where all elements on the plan $k = 3$ are equal to zero, then based on definition 3:

$$\det[A_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & 0 & 0 & 0 \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & 0 & 0 & 0 \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & 0 & 0 & 0 \end{array}\right)$$

$$= a_{111} \cdot a_{222} \cdot 0 - a_{111} \cdot a_{232} \cdot 0 - a_{111} \cdot 0 \cdot a_{332} + a_{111} \cdot 0 \cdot a_{322}$$

$$-a_{112} \cdot a_{221} \cdot 0 + a_{112} \cdot 0 \cdot a_{331} + a_{112} \cdot a_{231} \cdot 0 - a_{112} \cdot 0 \cdot a_{321}$$

$$+0 \cdot a_{221} \cdot a_{332} - 0 \cdot a_{222} \cdot a_{331} - 0 \cdot a_{231} \cdot a_{322} + 0 \cdot a_{232} \cdot a_{321}$$

$$-a_{121} \cdot a_{212} \cdot 0 + a_{121} \cdot 0 \cdot a_{332} + a_{121} \cdot a_{232} \cdot 0 - a_{121} \cdot 0 \cdot a_{312}$$

$$+a_{122} \cdot a_{211} \cdot 0 - a_{122} \cdot 0 \cdot a_{331} - a_{122} \cdot a_{231} \cdot 0 + a_{122} \cdot 0 \cdot a_{311}$$

$$-0 \cdot a_{211} \cdot a_{332} + 0 \cdot a_{212} \cdot a_{331} + 0 \cdot a_{231} \cdot a_{312} - 0 \cdot a_{232} \cdot a_{311}$$

$$+a_{131} \cdot a_{212} \cdot 0 - a_{131} \cdot 0 \cdot a_{322} - a_{131} \cdot a_{222} \cdot 0 + a_{131} \cdot 0 \cdot a_{312}$$

$$-a_{132} \cdot a_{211} \cdot 0 + a_{132} \cdot 0 \cdot a_{321} + a_{132} \cdot a_{221} \cdot 0 - a_{132} \cdot 0 \cdot a_{311}$$

$$+0 \cdot a_{211} \cdot a_{322} - 0 \cdot a_{212} \cdot a_{321} - 0 \cdot a_{221} \cdot a_{312} + 0 \cdot a_{222} \cdot a_{311} = 0.$$

Based on definition 2 and definition 3, we can see that each term is multiplied with elements of each plan once, hence the proof can be easily seen, as presented above.

Proposition 2 *Suppose that A is cubic-matrix of order 2 or 3, and let's be B the cubic-matrix of same order with A, which obtained from A by multiplying any single: "horizontal layer" or "vertical page" or "vertical layer" with scalar α. Then $det(B) = \alpha \cdot det(A)$, if $\alpha \neq 0$.*

Proof. We are discussing the following cases for cubic matrix of order 2 and cases for cubic matrix of order 3:

Case 1. The cubic-matrix A of order 2, (and B has order 2), we will proof the case 1 for each "horizontal layer", "vertical page" and "vertical layer", as following:

1. For plan $i = 1$: Let A be cubic-matrix of order 2, where all elements on the plan $i = 1$ are equal to zero, then based on definition 2:

$$\det[B_{2\times2\times2}] = \det\begin{pmatrix} \alpha \cdot a_{111} & \alpha \cdot a_{121} & \alpha \cdot a_{112} & \alpha \cdot a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{pmatrix}$$

$$= \alpha \cdot a_{111} \cdot a_{222} - \alpha \cdot a_{112} \cdot a_{221} - \alpha \cdot a_{121} \cdot a_{212} + \alpha \cdot a_{122} \cdot a_{211} = \alpha \cdot \det[A_{2\times2\times2}]$$

2. For plan $i = 2$: Let A be cubic-matrix of order 2, where all elements on the plan $i = 2$ are equal to zero, then based on definition 2:

$$\det[B_{2\times2\times2}] = \det\begin{pmatrix} a_{111} & a_{121} & a_{112} & a_{122} \\ \alpha \cdot a_{211} & \alpha \cdot a_{221} & \alpha \cdot a_{212} & \alpha \cdot a_{222} \end{pmatrix}$$

$$= a_{111} \cdot \alpha \cdot a_{222} - a_{112} \cdot \alpha \cdot a_{221} - a_{121} \cdot \alpha \cdot a_{212} + a_{122} \cdot \alpha \cdot a_{211} = \alpha \cdot \det[A_{2\times2\times2}]$$

3. For plan $j = 1$: Let A be cubic-matrix of order 2, where all elements on the plan $j = 1$ are equal to zero, then based on definition 2:

$$\det[B_{2\times2\times2}] = \det\begin{pmatrix} \alpha \cdot a_{111} & a_{121} & \alpha \cdot a_{112} & a_{122} \\ \alpha \cdot a_{211} & a_{221} & \alpha \cdot a_{212} & a_{222} \end{pmatrix}$$

$$= \alpha \cdot a_{111} \cdot a_{222} - \alpha \cdot a_{112} \cdot a_{221} - a_{121} \cdot \alpha \cdot a_{212} + a_{122} \cdot \alpha \cdot a_{211} = \alpha \cdot \det[A_{2\times2\times2}]$$

4. For plan $j = 2$: Let A be cubic-matrix of order 2, where all elements on the plan $j = 2$ are equal to zero, then based on definition 2:

$$\det[B_{2\times2\times2}] = \det\begin{pmatrix} a_{111} & \alpha \cdot a_{121} & a_{112} & \alpha \cdot a_{122} \\ a_{211} & \alpha \cdot a_{221} & a_{212} & \alpha \cdot a_{222} \end{pmatrix}$$

$$= a_{111} \cdot \alpha \cdot a_{222} - a_{112} \cdot \alpha \cdot a_{221} - \alpha \cdot a_{121} \cdot a_{212} + \alpha \cdot a_{122} \cdot a_{211} = \alpha \cdot \det[A_{2\times2\times2}]$$

5. For plan $k = 1$: Let A be cubic-matrix of order 2, where all elements on the plan $k = 1$ are equal to zero, then based on definition 2:

$$\det[B_{2\times2\times2}] = \det\begin{pmatrix} \alpha \cdot a_{111} & \alpha \cdot a_{121} & a_{112} & a_{122} \\ \alpha \cdot a_{211} & \alpha \cdot a_{221} & a_{212} & a_{222} \end{pmatrix}$$

$$= \alpha \cdot a_{a_{111}} \cdot a_{222} - a_{112} \cdot \alpha \cdot a_{221} - \alpha \cdot a_{121} \cdot a_{212} + a_{122} \cdot \alpha \cdot a_{211} = \alpha \cdot \det[A_{2\times2\times2}]$$

6. For plan $k = 2$: Let A be cubic-matrix of order 2, where all elements on the plan $k = 2$ are equal to zero, then based on definition 2:

$$\det[B_{2\times2\times2}] = \det\begin{pmatrix} a_{111} & a_{121} & \alpha \cdot a_{112} & \alpha \cdot a_{122} \\ a_{211} & a_{221} & \alpha \cdot a_{212} & \alpha \cdot a_{222} \end{pmatrix}$$

$$= a_{111} \cdot \alpha \cdot a_{222} - \alpha \cdot a_{112} \cdot a_{221} - a_{121} \cdot \alpha \cdot a_{212} + \alpha \cdot a_{122} \cdot a_{211} = \alpha \cdot \det[A_{2\times2\times2}]$$

Case 2. The cubic-matrix A of order 3, (and B has order 3)

1. For plan $i = 1$: Let A be cubic-matrix of order 2, where all elements on the plan $i = 1$ are equal to zero, then based on definition 3:

$$\det[B_{3\times3\times3}] = \det\left(\begin{matrix} \alpha \cdot a_{111} & \alpha \cdot a_{121} & \alpha \cdot a_{131} \\ a_{211} & a_{221} & a_{231} \\ a_{311} & a_{321} & a_{331} \end{matrix}\middle| \begin{matrix} \alpha \cdot a_{112} & \alpha \cdot a_{122} & \alpha \cdot a_{132} \\ a_{212} & a_{222} & a_{232} \\ a_{312} & a_{322} & a_{332} \end{matrix}\middle| \begin{matrix} \alpha \cdot a_{113} & \alpha \cdot a_{123} & \alpha \cdot a_{133} \\ a_{213} & a_{223} & a_{233} \\ a_{313} & a_{323} & a_{333} \end{matrix}\right)$$

$$= \alpha \cdot a_{111} \cdot a_{222} \cdot a_{333} - \alpha \cdot a_{111} \cdot a_{232} \cdot a_{323} - \alpha \cdot a_{111} \cdot a_{223} \cdot a_{332} + \alpha \cdot a_{111} \cdot a_{233} \cdot a_{322}$$

$$-\alpha \cdot a_{112} \cdot a_{221} \cdot a_{333} + \alpha \cdot a_{112} \cdot a_{223} \cdot a_{331} + \alpha \cdot a_{112} \cdot a_{231} \cdot a_{323} - \alpha \cdot a_{112} \cdot a_{233} \cdot a_{321}$$

$$+\alpha \cdot a_{113} \cdot a_{221} \cdot a_{332} - \alpha \cdot a_{113} \cdot a_{222} \cdot a_{331} - \alpha \cdot a_{113} \cdot a_{231} \cdot a_{322} + \alpha \cdot a_{113} \cdot a_{232} \cdot a_{321}$$

$$-\alpha \cdot a_{121} \cdot a_{212} \cdot a_{333} + \alpha \cdot a_{121} \cdot a_{213} \cdot a_{332} + \alpha \cdot a_{121} \cdot a_{232} \cdot a_{313} - \alpha \cdot a_{121} \cdot a_{233} \cdot a_{312}$$

$$+\alpha \cdot a_{122} \cdot a_{211} \cdot a_{333} - \alpha \cdot a_{122} \cdot a_{213} \cdot a_{331} - \alpha \cdot a_{122} \cdot a_{231} \cdot a_{313} + \alpha \cdot a_{122} \cdot a_{233} \cdot a_{311}$$

$$-\alpha \cdot a_{123} \cdot a_{211} \cdot a_{332} + \alpha \cdot a_{123} \cdot a_{212} \cdot a_{331} + \alpha \cdot a_{123} \cdot a_{231} \cdot a_{312} - \alpha \cdot a_{123} \cdot a_{232} \cdot a_{311}$$

$$+\alpha \cdot a_{131} \cdot a_{212} \cdot a_{323} - \alpha \cdot a_{131} \cdot a_{213} \cdot a_{322} - \alpha \cdot a_{131} \cdot a_{222} \cdot a_{313} + \alpha \cdot a_{131} \cdot a_{223} \cdot a_{312}$$

$$-\alpha \cdot a_{132} \cdot a_{211} \cdot a_{323} + \alpha \cdot a_{132} \cdot a_{213} \cdot a_{321} + \alpha \cdot a_{132} \cdot a_{221} \cdot a_{313} - \alpha \cdot a_{132} \cdot a_{223} \cdot a_{311}$$

$$+\alpha \cdot a_{133} \cdot a_{211} \cdot a_{322} - \alpha \cdot a_{133} \cdot a_{212} \cdot a_{321} - \alpha \cdot a_{133} \cdot a_{221} \cdot a_{312} + \alpha \cdot a_{133} \cdot a_{222} \cdot a_{311} = \alpha \cdot \det[A_{3\times3\times3}]$$

2. For plan $i = 2$: Let A be cubic-matrix of order 2, where all elements on the plan $i = 2$ are equal to zero, then based on definition 3:

$$\det[B_{3\times3\times3}] = \det\left(\begin{matrix} a_{111} & a_{121} & a_{131} \\ \alpha \cdot a_{211} & \alpha \cdot a_{221} & \alpha \cdot a_{231} \\ a_{311} & a_{321} & a_{331} \end{matrix}\middle| \begin{matrix} a_{112} & a_{122} & a_{132} \\ \alpha \cdot a_{212} & \alpha \cdot a_{222} & \alpha \cdot a_{232} \\ a_{312} & a_{322} & a_{332} \end{matrix}\middle| \begin{matrix} a_{113} & a_{123} & a_{133} \\ \alpha \cdot a_{213} & \alpha \cdot a_{223} & \alpha \cdot a_{233} \\ a_{313} & a_{323} & a_{333} \end{matrix}\right)$$

$$= a_{111} \cdot \alpha \cdot a_{222} \cdot a_{333} - a_{111} \cdot \alpha \cdot a_{232} \cdot a_{323} - a_{111} \cdot \alpha \cdot a_{223} \cdot a_{332} + a_{111} \cdot \alpha \cdot a_{233} \cdot a_{322}$$

$$-a_{112} \cdot \alpha \cdot a_{221} \cdot a_{333} + a_{112} \cdot \alpha \cdot a_{223} \cdot a_{331} + a_{112} \cdot \alpha \cdot a_{231} \cdot a_{323} - a_{112} \cdot \alpha \cdot a_{233} \cdot a_{321}$$

$$+a_{113} \cdot \alpha \cdot a_{221} \cdot a_{332} - a_{113} \cdot \alpha \cdot a_{222} \cdot a_{331} - a_{113} \cdot \alpha \cdot a_{231} \cdot a_{322} + a_{113} \cdot \alpha \cdot a_{232} \cdot a_{321}$$

$$-a_{121} \cdot \alpha \cdot a_{212} \cdot a_{333} + a_{121} \cdot \alpha \cdot a_{213} \cdot a_{332} + a_{121} \cdot \alpha \cdot a_{232} \cdot a_{313} - a_{121} \cdot \alpha \cdot a_{233} \cdot a_{312}$$

$$+a_{122} \cdot \alpha \cdot a_{211} \cdot a_{333} - a_{122} \cdot \alpha \cdot a_{213} \cdot a_{331} - a_{122} \cdot \alpha \cdot a_{231} \cdot a_{313} + a_{122} \cdot \alpha \cdot a_{233} \cdot a_{311}$$

$$-a_{123} \cdot \alpha \cdot a_{211} \cdot a_{332} + a_{123} \cdot \alpha \cdot a_{212} \cdot a_{331} + a_{123} \cdot \alpha \cdot a_{231} \cdot a_{312} - a_{123} \cdot \alpha \cdot a_{232} \cdot a_{311}$$

$$+a_{131} \cdot \alpha \cdot a_{212} \cdot a_{323} - a_{131} \cdot \alpha \cdot a_{213} \cdot a_{322} - a_{131} \cdot \alpha \cdot a_{222} \cdot a_{313} + a_{131} \cdot \alpha \cdot a_{223} \cdot a_{312}$$

$$-a_{132} \cdot \alpha \cdot a_{211} \cdot a_{323} + a_{132} \cdot \alpha \cdot a_{213} \cdot a_{321} + a_{132} \cdot \alpha \cdot a_{221} \cdot a_{313} - a_{132} \cdot \alpha \cdot a_{223} \cdot a_{311}$$

$$+a_{133} \cdot \alpha \cdot a_{211} \cdot a_{322} - a_{133} \cdot \alpha \cdot a_{212} \cdot a_{321} - a_{133} \cdot \alpha \cdot a_{221} \cdot a_{312} + a_{133} \cdot \alpha \cdot a_{222} \cdot a_{311} = \alpha \cdot \det[A_{3\times3\times3}]$$

3. For plan $i = 3$: Let A be cubic-matrix of order 2, where all elements on the plan $i = 3$ are equal to zero, then based on definition 3:

$$\det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ \alpha\cdot a_{311} & \alpha\cdot a_{321} & \alpha\cdot a_{331} & \alpha\cdot a_{312} & \alpha\cdot a_{322} & \alpha\cdot a_{332} & \alpha\cdot a_{313} & \alpha\cdot a_{323} & \alpha\cdot a_{333} \end{array}\right)$$

$$= a_{111}\cdot a_{222}\cdot\alpha\cdot a_{333} - a_{111}\cdot a_{232}\cdot\alpha\cdot a_{323} - a_{111}\cdot a_{223}\cdot\alpha\cdot a_{332} + a_{111}\cdot a_{233}\cdot\alpha\cdot a_{322}$$

$$-a_{112}\cdot a_{221}\cdot\alpha\cdot a_{333} + a_{112}\cdot a_{223}\cdot\alpha\cdot a_{331} + a_{112}\cdot a_{231}\cdot\alpha\cdot a_{323} - a_{112}\cdot a_{233}\cdot\alpha\cdot a_{321}$$

$$+a_{113}\cdot a_{221}\cdot\alpha\cdot a_{332} - a_{113}\cdot a_{222}\cdot\alpha\cdot a_{331} - a_{113}\cdot a_{231}\cdot\alpha\cdot a_{322} + a_{113}\cdot a_{232}\cdot\alpha\cdot a_{321}$$

$$-a_{121}\cdot a_{212}\cdot\alpha\cdot a_{333} + a_{121}\cdot a_{213}\cdot\alpha\cdot a_{332} + a_{121}\cdot a_{232}\cdot\alpha\cdot a_{313} - a_{121}\cdot a_{233}\cdot\alpha\cdot a_{312}$$

$$+a_{122}\cdot a_{211}\cdot\alpha\cdot a_{333} - a_{122}\cdot a_{213}\cdot\alpha\cdot a_{331} - a_{122}\cdot a_{231}\cdot\alpha\cdot a_{313} + a_{122}\cdot a_{233}\cdot\alpha\cdot a_{311}$$

$$-a_{123}\cdot a_{211}\cdot\alpha\cdot a_{332} + a_{123}\cdot a_{212}\cdot\alpha\cdot a_{331} + a_{123}\cdot a_{231}\cdot\alpha\cdot a_{312} - a_{123}\cdot a_{232}\cdot\alpha\cdot a_{311}$$

$$+a_{131}\cdot a_{212}\cdot\alpha\cdot a_{323} - a_{131}\cdot a_{213}\cdot\alpha\cdot a_{322} - a_{131}\cdot a_{222}\cdot\alpha\cdot a_{313} + a_{131}\cdot a_{223}\cdot\alpha\cdot a_{312}$$

$$-a_{132}\cdot a_{211}\cdot\alpha\cdot a_{323} + a_{132}\cdot a_{213}\cdot\alpha\cdot a_{321} + a_{132}\cdot a_{221}\cdot\alpha\cdot a_{313} - a_{132}\cdot a_{223}\cdot\alpha\cdot a_{311}$$

$$+a_{133}\cdot a_{211}\cdot\alpha\cdot a_{322} - a_{133}\cdot a_{212}\cdot\alpha\cdot a_{321} - a_{133}\cdot a_{221}\cdot\alpha\cdot a_{312} + a_{133}\cdot a_{222}\cdot\alpha\cdot a_{311} = \alpha\cdot\det[A_{3\times3\times3}]$$

4. For plan $j = 1$: Let A be cubic-matrix of order 2, where all elements on the plan $i = 1$ are equal to zero, then based on definition 3:

$$\det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} \alpha\cdot a_{111} & a_{121} & a_{131} & \alpha\cdot a_{112} & a_{122} & a_{132} & \alpha\cdot a_{113} & a_{123} & a_{133} \\ \alpha\cdot a_{211} & a_{221} & a_{231} & \alpha\cdot a_{212} & a_{222} & a_{232} & \alpha\cdot a_{213} & a_{223} & a_{233} \\ \alpha\cdot a_{311} & a_{321} & a_{331} & \alpha\cdot a_{312} & a_{322} & a_{332} & \alpha\cdot a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$= \alpha\cdot a_{111}\cdot a_{222}\cdot a_{333} - \alpha\cdot a_{111}\cdot a_{232}\cdot a_{323} - \alpha\cdot a_{111}\cdot a_{223}\cdot a_{332} + \alpha\cdot a_{111}\cdot a_{233}\cdot a_{322}$$

$$-\alpha\cdot a_{112}\cdot a_{221}\cdot a_{333} + \alpha\cdot a_{112}\cdot a_{223}\cdot a_{331} + \alpha\cdot a_{112}\cdot a_{231}\cdot a_{323} - \alpha\cdot a_{112}\cdot a_{233}\cdot a_{321}$$

$$+\alpha\cdot a_{113}\cdot a_{221}\cdot a_{332} - \alpha\cdot a_{113}\cdot a_{222}\cdot a_{331} - \alpha\cdot a_{113}\cdot a_{231}\cdot a_{322} + \alpha\cdot a_{113}\cdot a_{232}\cdot a_{321}$$

$$-a_{121}\cdot\alpha\cdot a_{212}\cdot a_{333} + a_{121}\cdot\alpha\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot\alpha\cdot a_{313} - a_{121}\cdot a_{233}\cdot\alpha\cdot a_{312}$$

$$+a_{122}\cdot\alpha\cdot a_{211}\cdot a_{333} - a_{122}\cdot\alpha\cdot a_{213}\cdot a_{331} - a_{122}\cdot a_{231}\cdot\alpha\cdot a_{313} + a_{122}\cdot a_{233}\cdot\alpha\cdot a_{311}$$

$$-a_{123}\cdot\alpha\cdot a_{211}\cdot a_{332} + a_{123}\cdot\alpha\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot\alpha\cdot a_{312} - a_{123}\cdot a_{232}\cdot\alpha\cdot a_{311}$$

$$+a_{131}\cdot\alpha\cdot a_{212}\cdot a_{323} - a_{131}\cdot\alpha\cdot a_{213}\cdot a_{322} - a_{131}\cdot a_{222}\cdot\alpha\cdot a_{313} + a_{131}\cdot a_{223}\cdot\alpha\cdot a_{312}$$

$$-a_{132}\cdot\alpha\cdot a_{211}\cdot a_{323} + a_{132}\cdot\alpha\cdot a_{213}\cdot a_{321} + a_{132}\cdot a_{221}\cdot\alpha\cdot a_{313} - a_{132}\cdot a_{223}\cdot\alpha\cdot a_{311}$$

$$+a_{133}\cdot\alpha\cdot a_{211}\cdot a_{322}\,a_{133}\cdot\alpha\cdot a_{212}\cdot a_{321} - a_{133}\cdot a_{221}\cdot\alpha\cdot a_{312} + a_{133}\cdot a_{222}\cdot\alpha\cdot a_{311} = \alpha\cdot\det[A_{3\times3\times3}]$$

5. For plan $j = 2$: Let A be cubic-matrix of order 2, where all elements on the plan $i = 2$ are equal to zero, then based on definition 3:

$$\det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{111} & \alpha\cdot a_{121} & a_{131} & a_{112} & \alpha\cdot a_{122} & a_{132} & a_{113} & \alpha\cdot a_{123} & a_{133} \\ a_{211} & \alpha\cdot a_{221} & a_{231} & a_{212} & \alpha\cdot a_{222} & a_{232} & a_{213} & \alpha\cdot a_{223} & a_{233} \\ a_{311} & \alpha\cdot a_{321} & a_{331} & a_{312} & \alpha\cdot a_{322} & a_{332} & a_{313} & \alpha\cdot a_{323} & a_{333} \end{array}\right)$$

$$= a_{111}\cdot\alpha\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot\alpha\cdot a_{323} - a_{111}\cdot\alpha\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot\alpha\cdot a_{322}$$

$$-a_{112}\cdot\alpha\cdot a_{221}\cdot a_{333}+a_{112}\cdot\alpha\cdot a_{223}\cdot a_{331}+a_{112}\cdot a_{231}\cdot\alpha\cdot a_{323}-a_{112}\cdot a_{233}\cdot\alpha\cdot a_{321}$$

$$+a_{113}\cdot\alpha\cdot a_{221}\cdot a_{332}-a_{113}\cdot\alpha\cdot a_{222}\cdot a_{331}-a_{113}\cdot a_{231}\cdot\alpha\cdot a_{322}+a_{113}\cdot a_{232}\cdot\alpha\cdot a_{321}$$

$$-\alpha\cdot a_{121}\cdot a_{212}\cdot a_{333}+\alpha\cdot a_{121}\cdot a_{213}\cdot a_{332}+\alpha\cdot a_{121}\cdot a_{232}\cdot a_{313}-\alpha\cdot a_{121}\cdot a_{233}\cdot a_{312}$$

$$+\alpha\cdot a_{122}\cdot a_{211}\cdot a_{333}-\alpha\cdot a_{122}\cdot a_{213}\cdot a_{331}-\alpha\cdot a_{122}\cdot a_{231}\cdot a_{313}+\alpha\cdot a_{122}\cdot a_{233}\cdot a_{311}$$

$$-\alpha\cdot a_{123}\cdot a_{211}\cdot a_{332}+\alpha\cdot a_{123}\cdot a_{212}\cdot a_{331}+\alpha\cdot a_{123}\cdot a_{231}\cdot a_{312}-\alpha\cdot a_{123}\cdot a_{232}\cdot a_{311}$$

$$+a_{131}\cdot a_{212}\cdot\alpha\cdot a_{323}-a_{131}\cdot a_{213}\cdot\alpha\cdot a_{322}-a_{131}\cdot\alpha\cdot a_{222}\cdot a_{313}+a_{131}\cdot\alpha\cdot a_{223}\cdot a_{312}$$

$$-a_{132}\cdot a_{211}\cdot\alpha\cdot a_{323}+a_{132}\cdot a_{213}\cdot\alpha\cdot a_{321}+a_{132}\cdot\alpha\cdot a_{221}\cdot a_{313}-a_{132}\cdot\alpha\cdot a_{223}\cdot a_{311}$$

$$+a_{133}\cdot a_{211}\cdot\alpha\cdot a_{322}-a_{133}\cdot a_{212}\cdot\alpha\cdot a_{321}-a_{133}\cdot\alpha\cdot a_{221}\cdot a_{312}+a_{133}\cdot\alpha\cdot a_{222}\cdot a_{311}=\alpha\cdot\det[A_{3\times3\times3}]$$

6. For plan $j=3$: Let A be cubic-matrix of order 2, where all elements on the plan $i=3$ are equal to zero, then based on definition 3:

$$\det[B_{3\times3\times3}]=\det\begin{pmatrix}a_{111} & a_{121} & \alpha\cdot a_{131} & a_{112} & a_{122} & \alpha\cdot a_{132} & a_{113} & a_{123} & \alpha\cdot a_{133}\\ a_{211} & a_{221} & \alpha\cdot a_{231} & a_{212} & a_{222} & \alpha\cdot a_{232} & a_{213} & a_{223} & \alpha\cdot a_{233}\\ a_{311} & a_{321} & \alpha\cdot a_{331} & a_{312} & a_{322} & \alpha\cdot a_{332} & a_{313} & a_{323} & \alpha\cdot a_{333}\end{pmatrix}$$

$$=a_{111}\cdot a_{222}\cdot\alpha\cdot a_{333}-a_{111}\cdot\alpha\cdot a_{232}\cdot a_{323}-a_{111}\cdot a_{223}\cdot\alpha\cdot a_{332}+a_{111}\cdot\alpha\cdot a_{233}\cdot a_{322}$$

$$-a_{112}\cdot a_{221}\cdot\alpha\cdot a_{333}+a_{112}\cdot a_{223}\cdot\alpha\cdot a_{331}+a_{112}\cdot\alpha\cdot a_{231}\cdot a_{323}-a_{112}\cdot\alpha\cdot a_{233}\cdot a_{321}$$

$$+a_{113}\cdot a_{221}\cdot\alpha\cdot a_{332}-a_{113}\cdot a_{222}\cdot\alpha\cdot a_{331}-a_{113}\cdot\alpha\cdot a_{231}\cdot a_{322}+a_{113}\cdot\alpha\cdot a_{232}\cdot a_{321}$$

$$-a_{121}\cdot a_{212}\cdot\alpha\cdot a_{333}+a_{121}\cdot a_{213}\cdot\alpha\cdot a_{332}+a_{121}\cdot\alpha\cdot a_{232}\cdot a_{313}-a_{121}\cdot\alpha\cdot a_{233}\cdot a_{312}$$

$$+a_{122}\cdot a_{211}\cdot\alpha\cdot a_{333}-a_{122}\cdot a_{213}\cdot\alpha\cdot a_{331}-a_{122}\cdot\alpha\cdot a_{231}\cdot a_{313}+a_{122}\cdot\alpha\cdot a_{233}\cdot a_{311}$$

$$-a_{123}\cdot a_{211}\cdot\alpha\cdot a_{332}+a_{123}\cdot a_{212}\cdot\alpha\cdot a_{331}+a_{123}\cdot\alpha\cdot a_{231}\cdot a_{312}-a_{123}\cdot\alpha\cdot a_{232}\cdot a_{311}$$

$$+\alpha\cdot a_{131}\cdot a_{212}\cdot a_{323}-\alpha\cdot a_{131}\cdot a_{213}\cdot a_{322}-\alpha\cdot a_{131}\cdot a_{222}\cdot a_{313}+\alpha\cdot a_{131}\cdot a_{223}\cdot a_{312}$$

$$-\alpha\cdot a_{132}\cdot a_{211}\cdot a_{323}+\alpha\cdot a_{132}\cdot a_{213}\cdot a_{321}+\alpha\cdot a_{132}\cdot a_{221}\cdot a_{313}-\alpha\cdot a_{132}\cdot a_{223}\cdot a_{311}$$

$$+\alpha\cdot a_{133}\cdot a_{211}\cdot a_{322}-\alpha\cdot a_{133}\cdot a_{212}\cdot a_{321}-\alpha\cdot a_{133}\cdot a_{221}\cdot a_{312}+\alpha\cdot a_{133}\cdot a_{222}\cdot a_{311}=\alpha\cdot\det[A_{3\times3\times3}]$$

7. For plan $k=1$: Let A be cubic-matrix of order 2, where all elements on the plan $k=1$ are equal to zero, then based on definition 3:

$$\det[B_{3\times3\times3}]=\det\begin{pmatrix}\alpha\cdot a_{111} & \alpha\cdot a_{121} & \alpha\cdot a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133}\\ \alpha\cdot a_{211} & \alpha\cdot a_{221} & \alpha\cdot a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233}\\ \alpha\cdot a_{311} & \alpha\cdot a_{321} & \alpha\cdot a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333}\end{pmatrix}$$

$$=\alpha\cdot a_{111}\cdot a_{222}\cdot a_{333}-\alpha\cdot a_{111}\cdot a_{232}\cdot a_{323}-\alpha\cdot a_{111}\cdot a_{223}\cdot a_{332}+\alpha\cdot a_{111}\cdot a_{233}\cdot a_{322}$$

$$-a_{112}\cdot\alpha\cdot a_{221}\cdot a_{333}+a_{112}\cdot a_{223}\cdot\alpha\cdot a_{331}+a_{112}\cdot\alpha\cdot a_{231}\cdot a_{323}-a_{112}\cdot a_{233}\cdot\alpha\cdot a_{321}$$

$$+a_{113}\cdot\alpha\cdot a_{221}\cdot a_{332}-a_{113}\cdot a_{222}\cdot\alpha\cdot a_{331}-a_{113}\cdot\alpha\cdot a_{231}\cdot a_{322}+a_{113}\cdot a_{232}\cdot\alpha\cdot a_{321}$$

$$-\alpha\cdot a_{121}\cdot a_{212}\cdot a_{333}+\alpha\cdot a_{121}\cdot a_{213}\cdot a_{332}+\alpha\cdot a_{121}\cdot a_{232}\cdot a_{313}-\alpha\cdot a_{121}\cdot a_{233}\cdot a_{312}$$

$$+a_{122}\cdot\alpha\cdot a_{211}\cdot a_{333}-a_{122}\cdot a_{213}\cdot\alpha\cdot a_{331}-a_{122}\cdot\alpha\cdot a_{231}\cdot a_{313}+a_{122}\cdot a_{233}\cdot\alpha\cdot a_{311}$$

$$-a_{123}\cdot\alpha\cdot a_{211}\cdot a_{332}+a_{123}\cdot a_{212}\cdot\alpha\cdot a_{331}+a_{123}\cdot\alpha\cdot a_{231}\cdot a_{312}-a_{123}\cdot a_{232}\cdot\alpha\cdot a_{311}$$

$$+\alpha\cdot a_{131}\cdot a_{212}\cdot a_{323}-\alpha\cdot a_{131}\cdot a_{213}\cdot a_{322}-\alpha\cdot a_{131}\cdot a_{222}\cdot a_{313}+\alpha\cdot a_{131}\cdot a_{223}\cdot a_{312}$$

$$-a_{132}\cdot\alpha\cdot a_{211}\cdot a_{323}+a_{132}\cdot a_{213}\cdot\alpha\cdot a_{321}+a_{132}\cdot\alpha\cdot a_{221}\cdot a_{313}-a_{132}\cdot a_{223}\cdot\alpha\cdot a_{311}$$

$$a_{133}\cdot\alpha\cdot a_{211}\cdot a_{322}-a_{133}\cdot a_{212}\cdot\alpha\cdot a_{321}-a_{133}\cdot\alpha\cdot a_{221}\cdot a_{312}+a_{133}\cdot a_{222}\cdot\alpha\cdot a_{311}=\alpha\cdot\det[A_{3\times3\times3}]$$

8. For plan $k=2$: Let A be cubic-matrix of order 2, where all elements on the plan $k=2$ are equal to zero, then based on definition 3:

$$\det[B_{3\times3\times3}]=\det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & \alpha\cdot a_{112} & \alpha\cdot a_{122} & \alpha\cdot a_{132} & a_{113} & a_{123} & a_{133}\\ a_{211} & a_{221} & a_{231} & \alpha\cdot a_{212} & \alpha\cdot a_{222} & \alpha\cdot a_{232} & a_{213} & a_{223} & a_{233}\\ a_{311} & a_{321} & a_{331} & \alpha\cdot a_{312} & \alpha\cdot a_{322} & \alpha\cdot a_{332} & a_{313} & a_{323} & a_{333}\end{array}\right)$$

$$=a_{111}\cdot\alpha\cdot a_{222}\cdot a_{333}-a_{111}\cdot\alpha\cdot a_{232}\cdot a_{323}-a_{111}\cdot a_{223}\cdot\alpha\cdot a_{332}+a_{111}\cdot a_{233}\cdot\alpha\cdot a_{322}$$

$$-\alpha\cdot a_{112}\cdot a_{221}\cdot a_{333}+\alpha\cdot a_{112}\cdot a_{223}\cdot a_{331}+\alpha\cdot a_{112}\cdot a_{231}\cdot a_{323}-\alpha\cdot a_{112}\cdot a_{233}\cdot a_{321}$$

$$+a_{113}\cdot a_{221}\cdot\alpha\cdot a_{332}-a_{113}\cdot\alpha\cdot a_{222}\cdot a_{331}-a_{113}\cdot a_{231}\cdot\alpha\cdot a_{322}+a_{113}\cdot\alpha\cdot a_{232}\cdot a_{321}$$

$$-a_{121}\cdot\alpha\cdot a_{212}\cdot a_{333}+a_{121}\cdot a_{213}\cdot\alpha\cdot a_{332}+a_{121}\cdot\alpha\cdot a_{232}\cdot a_{313}-a_{121}\cdot a_{233}\cdot\alpha\cdot a_{312}$$

$$+\alpha\cdot a_{122}\cdot a_{211}\cdot a_{333}-\alpha\cdot a_{122}\cdot a_{213}\cdot a_{331}-\alpha\cdot a_{122}\cdot a_{231}\cdot a_{313}+\alpha\cdot a_{122}\cdot a_{233}\cdot a_{311}$$

$$-a_{123}\cdot a_{211}\cdot\alpha\cdot a_{332}+a_{123}\cdot\alpha\cdot a_{212}\cdot a_{331}+a_{123}\cdot a_{231}\cdot\alpha\cdot a_{312}-a_{123}\cdot\alpha\cdot a_{232}\cdot a_{311}$$

$$+a_{131}\cdot\alpha\cdot a_{212}\cdot a_{323}-a_{131}\cdot a_{213}\cdot\alpha\cdot a_{322}-a_{131}\cdot\alpha\cdot a_{222}\cdot a_{313}+a_{131}\cdot a_{223}\cdot\alpha\cdot a_{312}$$

$$-\alpha\cdot a_{132}\cdot a_{211}\cdot a_{323}+\alpha\cdot a_{132}\cdot a_{213}\cdot a_{321}+\alpha\cdot a_{132}\cdot a_{221}\cdot a_{313}-\alpha\cdot a_{132}\cdot a_{223}\cdot a_{311}$$

$$+a_{133}\cdot a_{211}\cdot\alpha\cdot a_{322}-a_{133}\cdot\alpha\cdot a_{212}\cdot a_{321}-a_{133}\cdot a_{221}\cdot\alpha\cdot a_{312}+a_{133}\cdot\alpha\cdot a_{222}\cdot a_{311}=\alpha\cdot\det[A_{3\times3\times3}]$$

9. For plan $k=3$: Let A be cubic-matrix of order 2, where all elements on the plan $k=3$ are equal to zero, then based on definition 3:

$$\det[B_{3\times3\times3}]=\det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & \alpha\cdot a_{113} & \alpha\cdot a_{123} & \alpha\cdot a_{133}\\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & \alpha\cdot a_{213} & \alpha\cdot a_{223} & \alpha\cdot a_{233}\\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & \alpha\cdot a_{313} & \alpha\cdot a_{323} & \alpha\cdot a_{333}\end{array}\right)$$

$$=a_{111}\cdot a_{222}\cdot\alpha\cdot a_{333}-a_{111}\cdot a_{232}\cdot\alpha\cdot a_{323}-a_{111}\cdot\alpha\cdot a_{223}\cdot a_{332}+a_{111}\cdot\alpha\cdot a_{233}\cdot a_{322}$$

$$-a_{112}\cdot a_{221}\cdot\alpha\cdot a_{333}+a_{112}\cdot\alpha\cdot a_{223}\cdot a_{331}+a_{112}\cdot a_{231}\cdot\alpha\cdot a_{323}-a_{112}\cdot\alpha\cdot a_{233}\cdot a_{321}$$

$$+\alpha\cdot a_{113}\cdot a_{221}\cdot a_{332}-\alpha\cdot a_{113}\cdot a_{222}\cdot a_{331}-\alpha\cdot a_{113}\cdot a_{231}\cdot a_{322}+\alpha\cdot a_{113}\cdot a_{232}\cdot a_{321}$$

$$-a_{121}\cdot a_{212}\cdot\alpha\cdot a_{333}+a_{121}\cdot\alpha\cdot a_{213}\cdot a_{332}+a_{121}\cdot a_{232}\cdot\alpha\cdot a_{313}-a_{121}\cdot\alpha\cdot a_{233}\cdot a_{312}$$

$$+a_{122}\cdot a_{211}\cdot\alpha\cdot a_{333}-a_{122}\cdot\alpha\cdot a_{213}\cdot a_{331}-a_{122}\cdot a_{231}\cdot\alpha\cdot a_{313}+a_{122}\cdot\alpha\cdot a_{233}\cdot a_{311}$$

$$-\alpha\cdot a_{123}\cdot a_{211}\cdot a_{332}+\alpha\cdot a_{123}\cdot a_{212}\cdot a_{331}+\alpha\cdot a_{123}\cdot a_{231}\cdot a_{312}-\alpha\cdot a_{123}\cdot a_{232}\cdot a_{311}$$

$$+a_{131}\cdot a_{212}\cdot\alpha\cdot a_{323}-a_{131}\cdot\alpha\cdot a_{213}\cdot a_{322}-a_{131}\cdot a_{222}\cdot\alpha\cdot a_{313}+a_{131}\cdot\alpha\cdot a_{223}\cdot a_{312}$$

$$-a_{132} \cdot a_{211} \cdot \alpha \cdot a_{323} + a_{132} \cdot \alpha \cdot a_{213} \cdot a_{321} + a_{132} \cdot a_{221} \cdot \alpha \cdot a_{313} - a_{132} \cdot \alpha \cdot a_{223} \cdot a_{311}$$

$$+\alpha \cdot a_{133} \cdot a_{211} \cdot a_{322} - \alpha \cdot a_{133} \cdot a_{212} \cdot a_{321} - \alpha \cdot a_{133} \cdot a_{221} \cdot a_{312} + \alpha \cdot a_{133} \cdot a_{222} \cdot a_{311} = \alpha \cdot \det[A_{3\times3\times3}]$$

Example 3 *Let's be A a cubic matrix with order 2 then we will obtain determinant a cubic-matrix B from A by multiplying any plan $i = 1$ with scalar 3 and have,*

$$\det[B_{2\times2\times2}] = \det\begin{pmatrix} 3\cdot2 & 3\cdot1 & 3\cdot4 & 3\cdot7 \\ 3 & 5 & 3 & 2 \end{pmatrix} = 3\cdot2\cdot2 - 3\cdot4\cdot5 - 3\cdot1\cdot3 + 3\cdot7\cdot3 = 6.$$

If we compare with example 2, we can see that $|A| = 3 \cdot |B|$.

Theorem 2 *Let's be A a cubic-matrix of order 2 or order 3, and B be another cubic-matrix, which obtained from A by interchanging the location of two consecutive "horizontal layer" in i-index, then $det(A) = det(B)$.*

Proof. We try the cases in order for cubic matrix of order 2 and cases for cubic matrix of order 3:

Case 1. Let B be cubic-matrix of order 2, where we have interchanged to "horizontal layer", then based on definition 2: Let B be cubic-matrix of order 2, where we have interchanged two "horizontal layer", then based on definition 2:

$$\det[B_{2\times2\times2}] = \det\begin{pmatrix} a_{211} & a_{221} & a_{212} & a_{222} \\ a_{111} & a_{121} & a_{112} & a_{122} \end{pmatrix}$$

$$= a_{211} \cdot a_{122} - a_{212} \cdot a_{121} - a_{221} \cdot a_{112} + a_{222} \cdot a_{111} = \det[A_{2\times2\times2}]$$

Case 2. Let B be cubic-matrix of order 3, where we have interchanged to "horizontal layer", then based on definition 2:

1. Let B be cubic-matrix of order 3, where we have interchanged two "horizontal layer" (first layer with the second layer), then based on definition 3:

$$\det[B_{3\times3\times3}] = \det\begin{pmatrix} a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{pmatrix}$$

$$= a_{211} \cdot a_{122} \cdot a_{333} - a_{211} \cdot a_{132} \cdot a_{323} - a_{211} \cdot a_{123} \cdot a_{332} + a_{211} \cdot a_{133} \cdot a_{322}$$

$$-a_{212} \cdot a_{121} \cdot a_{333} + a_{212} \cdot a_{123} \cdot a_{331} + a_{212} \cdot a_{131} \cdot a_{323} - a_{212} \cdot a_{133} \cdot a_{321}$$

$$+a_{213} \cdot a_{121} \cdot a_{332} - a_{213} \cdot a_{122} \cdot a_{331} - a_{213} \cdot a_{131} \cdot a_{322} + a_{213} \cdot a_{132} \cdot a_{321}$$

$$-a_{221} \cdot a_{112} \cdot a_{333} + a_{221} \cdot a_{113} \cdot a_{332} + a_{221} \cdot a_{132} \cdot a_{313} - a_{221} \cdot a_{133} \cdot a_{312}$$

$$+a_{222} \cdot a_{111} \cdot a_{333} - a_{222} \cdot a_{113} \cdot a_{331} - a_{222} \cdot a_{131} \cdot a_{313} + a_{222} \cdot a_{133} \cdot a_{311}$$

$$-a_{223} \cdot a_{111} \cdot a_{332} + a_{223} \cdot a_{112} \cdot a_{331} + a_{223} \cdot a_{131} \cdot a_{312} - a_{223} \cdot a_{132} \cdot a_{311}$$

$$+a_{231} \cdot a_{112} \cdot a_{323} - a_{231} \cdot a_{113} \cdot a_{322} - a_{231} \cdot a_{122} \cdot a_{313} + a_{231} \cdot a_{123} \cdot a_{312}$$

$$-a_{232} \cdot a_{111} \cdot a_{323} + a_{232} \cdot a_{113} \cdot a_{321} + a_{232} \cdot a_{121} \cdot a_{313} - a_{232} \cdot a_{123} \cdot a_{311}$$

$$+a_{233} \cdot a_{111} \cdot a_{322} - a_{233} \cdot a_{112} \cdot a_{311} - a_{233} \cdot a_{121} \cdot a_{312} + a_{233} \cdot a_{122} \cdot a_{311}$$

$$= a_{111} \cdot a_{222} \cdot a_{333} - a_{111} \cdot a_{232} \cdot a_{323} - a_{111} \cdot a_{223} \cdot a_{332} + a_{111} \cdot a_{233} \cdot a_{322}$$

$$-a_{112}\cdot a_{221}\cdot a_{333}+a_{112}\cdot a_{223}\cdot a_{331}+a_{112}\cdot a_{231}\cdot a_{323}-a_{112}\cdot a_{233}\cdot a_{321}$$

$$+a_{113}\cdot a_{221}\cdot a_{332}-a_{113}\cdot a_{222}\cdot a_{331}-a_{113}\cdot a_{231}\cdot a_{322}+a_{113}\cdot a_{232}\cdot a_{321}$$

$$-a_{121}\cdot a_{212}\cdot a_{333}+a_{121}\cdot a_{213}\cdot a_{332}+a_{121}\cdot a_{232}\cdot a_{313}-a_{121}\cdot a_{233}\cdot a_{312}$$

$$+a_{122}\cdot a_{211}\cdot a_{333}-a_{122}\cdot a_{213}\cdot a_{331}-a_{122}\cdot a_{231}\cdot a_{313}+a_{122}\cdot a_{233}\cdot a_{311}$$

$$-a_{123}\cdot a_{211}\cdot a_{332}+a_{123}\cdot a_{212}\cdot a_{331}+a_{123}\cdot a_{231}\cdot a_{312}-a_{123}\cdot a_{232}\cdot a_{311}$$

$$+a_{131}\cdot a_{212}\cdot a_{323}-a_{131}\cdot a_{213}\cdot a_{322}-a_{131}\cdot a_{222}\cdot a_{313}+a_{131}\cdot a_{223}\cdot a_{312}$$

$$-a_{132}\cdot a_{211}\cdot a_{323}+a_{132}\cdot a_{213}\cdot a_{321}+a_{132}\cdot a_{221}\cdot a_{313}-a_{132}\cdot a_{223}\cdot a_{311}$$

$$+a_{133}\cdot a_{211}\cdot a_{322}-a_{133}\cdot a_{212}\cdot a_{321}-a_{133}\cdot a_{221}\cdot a_{312}+a_{133}\cdot a_{222}\cdot a_{311}=\det[A_{3\times 3\times 3}]$$

2. Let B be cubic-matrix of order 3, where we have interchanged two "horizontal layer" (second layer with the third layer), then based on definition 3:

$$\det[B_{3\times 3\times 3}]=\det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \end{array}\right)$$

$$=a_{111}\cdot a_{322}\cdot a_{233}-a_{111}\cdot a_{332}\cdot a_{223}-a_{111}\cdot a_{323}\cdot a_{232}+a_{111}\cdot a_{333}\cdot a_{222}$$

$$-a_{112}\cdot a_{321}\cdot a_{233}+a_{112}\cdot a_{323}\cdot a_{231}+a_{112}\cdot a_{331}\cdot a_{223}-a_{112}\cdot a_{333}\cdot a_{221}$$

$$+a_{113}\cdot a_{321}\cdot a_{232}-a_{113}\cdot a_{322}\cdot a_{231}-a_{113}\cdot a_{331}\cdot a_{222}+a_{113}\cdot a_{332}\cdot a_{221}$$

$$-a_{121}\cdot a_{312}\cdot a_{233}+a_{121}\cdot a_{313}\cdot a_{232}+a_{121}\cdot a_{332}\cdot a_{213}-a_{121}\cdot a_{333}\cdot a_{212}$$

$$+a_{122}\cdot a_{311}\cdot a_{233}-a_{122}\cdot a_{313}\cdot a_{231}-a_{122}\cdot a_{331}\cdot a_{213}+a_{122}\cdot a_{333}\cdot a_{211}$$

$$-a_{123}\cdot a_{311}\cdot a_{232}+a_{123}\cdot a_{312}\cdot a_{231}+a_{123}\cdot a_{331}\cdot a_{212}-a_{123}\cdot a_{332}\cdot a_{211}$$

$$+a_{131}\cdot a_{312}\cdot a_{223}-a_{131}\cdot a_{313}\cdot a_{222}-a_{131}\cdot a_{322}\cdot a_{213}+a_{131}\cdot a_{323}\cdot a_{212}$$

$$-a_{132}\cdot a_{311}\cdot a_{223}+a_{132}\cdot a_{313}\cdot a_{221}+a_{132}\cdot a_{321}\cdot a_{213}-a_{132}\cdot a_{323}\cdot a_{211}$$

$$+a_{133}\cdot a_{311}\cdot a_{222}-a_{133}\cdot a_{312}\cdot a_{221}-a_{133}\cdot a_{321}\cdot a_{212}+a_{133}\cdot a_{322}\cdot a_{211}$$

$$=a_{111}\cdot a_{222}\cdot a_{333}-a_{111}\cdot a_{232}\cdot a_{323}-a_{111}\cdot a_{223}\cdot a_{332}+a_{111}\cdot a_{233}\cdot a_{322}$$

$$-a_{112}\cdot a_{221}\cdot a_{333}+a_{112}\cdot a_{223}\cdot a_{331}+a_{112}\cdot a_{231}\cdot a_{323}-a_{112}\cdot a_{233}\cdot a_{321}$$

$$+a_{113}\cdot a_{221}\cdot a_{332}-a_{113}\cdot a_{222}\cdot a_{331}-a_{113}\cdot a_{231}\cdot a_{322}+a_{113}\cdot a_{232}\cdot a_{321}$$

$$-a_{121}\cdot a_{212}\cdot a_{333}+a_{121}\cdot a_{213}\cdot a_{332}+a_{121}\cdot a_{232}\cdot a_{313}-a_{121}\cdot a_{233}\cdot a_{312}$$

$$+a_{122}\cdot a_{211}\cdot a_{333}-a_{122}\cdot a_{213}\cdot a_{331}-a_{122}\cdot a_{231}\cdot a_{313}+a_{122}\cdot a_{233}\cdot a_{311}$$

$$-a_{123}\cdot a_{211}\cdot a_{332}+a_{123}\cdot a_{212}\cdot a_{331}+a_{123}\cdot a_{231}\cdot a_{312}-a_{123}\cdot a_{232}\cdot a_{311}$$

$$+a_{131}\cdot a_{212}\cdot a_{323}-a_{131}\cdot a_{213}\cdot a_{322}-a_{131}\cdot a_{222}\cdot a_{313}+a_{131}\cdot a_{223}\cdot a_{312}$$

$$-a_{132}\cdot a_{211}\cdot a_{323}+a_{132}\cdot a_{213}\cdot a_{321}+a_{132}\cdot a_{221}\cdot a_{313}-a_{132}\cdot a_{223}\cdot a_{311}$$

$$+a_{133} \cdot a_{211} \cdot a_{322} - a_{133} \cdot a_{212} \cdot a_{321} - a_{133} \cdot a_{221} \cdot a_{312} + a_{133} \cdot a_{222} \cdot a_{311} = \det[A_{3\times3\times3}]$$

Example 4 *Let A be $2 \times 2 \times 2$ 3D determinant than we will obtain determinant B from A by interchanging location of two horizontal layer in i index:*

$$\det[A_{2\times2\times2}] = \det\begin{pmatrix} 2 & 1 & 4 & 7 \\ 3 & 5 & 3 & 2 \end{pmatrix} = 2 \cdot 2 - 4 \cdot 5 - 1 \cdot 3 + 7 \cdot 3 = 2.$$

Then:

$$\det[A_{2\times2\times2}] = \det\begin{pmatrix} 3 & 5 & 3 & 2 \\ 2 & 1 & 4 & 7 \end{pmatrix} = 3 \cdot 7 - 3 \cdot 1 - 5 \cdot 4 + 2 \cdot 2 = 2.$$

If we compare results with example 2, we can see that we have the same result.

Example 5 *Let A be a $3 \times 3 \times 3$ 3D determinant than we will obtain determinant B from A by interchanging location of two horizontal layer in i index:*

$$\det[A_{3\times3\times3}] = \det\begin{pmatrix} 1 & 4 & 2 & 3 & 1 & 3 & 2 & 1 & 0 \\ 2 & 0 & 0 & 5 & 1 & 3 & 0 & 1 & 0 \\ 0 & 4 & 2 & 3 & 2 & 0 & 2 & 1 & 0 \end{pmatrix} = -3 + 6 - 4 + 24 + 24 + 10 + 10 - 4 + 6 - 6 = 63$$

Then:

$$\det[B_{3\times3\times3}] = \det\begin{pmatrix} 2 & 0 & 0 & 5 & 1 & 3 & 0 & 1 & 0 \\ 1 & 4 & 2 & 3 & 1 & 3 & 2 & 1 & 0 \\ 0 & 4 & 2 & 3 & 2 & 0 & 2 & 1 & 0 \end{pmatrix} = 6 - 4 + 10 + 10 - 4 - 3 + 24 + 24 - 6 + 6 = 63$$

If we compare results with example 2, we can see that we have the same result.

Theorem 3 *Let's be A a cubic-matrix of order 2 or order 3, and B be another cubic-matrix, which obtained from A by interchanging the location of two consecutive: "vertical page" in j-index or "vertical layer" in k-index, then $det(A) = -det(B)$.*

Proof. We discuss the cases in order for cubic matrix of order 2 and order 3:

Case 1. The cubic-matrix A of order 2, (and B has order 2), we will proof the case 1 for each "vertical page" and "vertical layer", as following:

1. For interchanging the location of two consecutive "vertical page": Let B be cubic-matrix of order 2, where we have interchanged two "vertical page", then based on definition 2:

$$\det[B_{2\times2\times2}] = \det\begin{pmatrix} a_{121} & a_{111} & a_{122} & a_{112} \\ a_{221} & a_{211} & a_{222} & a_{212} \end{pmatrix}$$

$$= a_{121} \cdot a_{212} - a_{122} \cdot a_{211} - a_{111} \cdot a_{222} + a_{112} \cdot a_{221} = -\det[A_{2\times2\times2}]$$

2. For interchanging the location of two consecutive "vertical layer": Let B be cubic-matrix of order 2, where we have interchanged two "vertical layer", then based on definition 2:

$$\det[B_{2\times2\times2}] = \det\begin{pmatrix} a_{112} & a_{122} & a_{111} & a_{121} \\ a_{212} & a_{222} & a_{211} & a_{221} \end{pmatrix}$$

$$= a_{112} \cdot a_{221} - a_{111} \cdot a_{222} - a_{122} \cdot a_{211} + a_{121} \cdot a_{212} = -\det[A_{2\times2\times2}]$$

Case 2. The cubic-matrix A of order 3, (and B has order 3), we will proof the case 2 for each "vertical page" and "vertical layer", as following:

1. For interchanging the location of two consecutive "vertical page": Let B be cubic-matrix of order 3, where we have interchanged two consecutive "vertical page" (First page with second page), then based on definition 3:

$$\det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{121} & a_{111} & a_{131} & a_{122} & a_{112} & a_{132} & a_{123} & a_{113} & a_{133} \\ a_{221} & a_{211} & a_{231} & a_{222} & a_{212} & a_{232} & a_{223} & a_{213} & a_{233} \\ a_{321} & a_{311} & a_{331} & a_{322} & a_{312} & a_{332} & a_{323} & a_{313} & a_{333} \end{array}\right)$$

$$= a_{121}\cdot a_{212}\cdot a_{333} - a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{233}\cdot a_{312}$$

$$-a_{122}\cdot a_{211}\cdot a_{333} + a_{122}\cdot a_{213}\cdot a_{331} + a_{122}\cdot a_{231}\cdot a_{313} - a_{122}\cdot a_{233}\cdot a_{311}$$

$$+a_{123}\cdot a_{211}\cdot a_{332} - a_{123}\cdot a_{212}\cdot a_{331} - a_{123}\cdot a_{231}\cdot a_{312} + a_{123}\cdot a_{232}\cdot a_{311}$$

$$-a_{111}\cdot a_{222}\cdot a_{333} + a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{233}\cdot a_{322}$$

$$+a_{112}\cdot a_{221}\cdot a_{333} - a_{112}\cdot a_{223}\cdot a_{331} - a_{112}\cdot a_{231}\cdot a_{323} + a_{112}\cdot a_{233}\cdot a_{321}$$

$$-a_{113}\cdot a_{221}\cdot a_{332} + a_{113}\cdot a_{222}\cdot a_{331} + a_{113}\cdot a_{231}\cdot a_{322} - a_{113}\cdot a_{232}\cdot a_{321}$$

$$+a_{131}\cdot a_{222}\cdot a_{313} - a_{131}\cdot a_{223}\cdot a_{312} - a_{131}\cdot a_{212}\cdot a_{323} + a_{131}\cdot a_{213}\cdot a_{322}$$

$$-a_{132}\cdot a_{221}\cdot a_{313} + a_{132}\cdot a_{223}\cdot a_{311} + a_{132}\cdot a_{211}\cdot a_{323} - a_{132}\cdot a_{213}\cdot a_{321}$$

$$+a_{133}\cdot a_{221}\cdot a_{312} - a_{133}\cdot a_{222}\cdot a_{311} - a_{133}\cdot a_{211}\cdot a_{322} + a_{133}\cdot a_{212}\cdot a_{321}$$

$$= -a_{111}\cdot a_{222}\cdot a_{333} + a_{111}\cdot a_{232}\cdot a_{323} + a_{111}\cdot a_{223}\cdot a_{332} - a_{111}\cdot a_{233}\cdot a_{322}$$

$$+a_{112}\cdot a_{221}\cdot a_{333} - a_{112}\cdot a_{223}\cdot a_{331} - a_{112}\cdot a_{231}\cdot a_{323} + a_{112}\cdot a_{233}\cdot a_{321}$$

$$-a_{113}\cdot a_{221}\cdot a_{332} + a_{113}\cdot a_{222}\cdot a_{331} + a_{113}\cdot a_{231}\cdot a_{322} - a_{113}\cdot a_{232}\cdot a_{321}$$

$$+a_{121}\cdot a_{212}\cdot a_{333} - a_{121}\cdot a_{213}\cdot a_{332} - a_{121}\cdot a_{232}\cdot a_{313} + a_{121}\cdot a_{233}\cdot a_{312}$$

$$-a_{122}\cdot a_{211}\cdot a_{333} + a_{122}\cdot a_{213}\cdot a_{331} + a_{122}\cdot a_{231}\cdot a_{313} - a_{122}\cdot a_{233}\cdot a_{311}$$

$$+a_{123}\cdot a_{211}\cdot a_{332} - a_{123}\cdot a_{212}\cdot a_{331} - a_{123}\cdot a_{231}\cdot a_{312} + a_{123}\cdot a_{232}\cdot a_{311}$$

$$-a_{131}\cdot a_{212}\cdot a_{323} + a_{131}\cdot a_{213}\cdot a_{322} + a_{131}\cdot a_{222}\cdot a_{313} - a_{131}\cdot a_{223}\cdot a_{312}$$

$$+a_{132}\cdot a_{211}\cdot a_{323} - a_{132}\cdot a_{213}\cdot a_{321} - a_{132}\cdot a_{221}\cdot a_{313} + a_{132}\cdot a_{223}\cdot a_{311}$$

$$-a_{133}\cdot a_{211}\cdot a_{322} + a_{133}\cdot a_{212}\cdot a_{321} + a_{133}\cdot a_{221}\cdot a_{312} - a_{133}\cdot a_{222}\cdot a_{311}$$

$$= -(a_{111}\cdot a_{222}\cdot a_{333} - a_{111}\cdot a_{232}\cdot a_{323} - a_{111}\cdot a_{223}\cdot a_{332} + a_{111}\cdot a_{233}\cdot a_{322}$$

$$-a_{112}\cdot a_{221}\cdot a_{333} + a_{112}\cdot a_{223}\cdot a_{331} + a_{112}\cdot a_{231}\cdot a_{323} - a_{112}\cdot a_{233}\cdot a_{321}$$

$$+a_{113}\cdot a_{221}\cdot a_{332} - a_{113}\cdot a_{222}\cdot a_{331} - a_{113}\cdot a_{231}\cdot a_{322} + a_{113}\cdot a_{232}\cdot a_{321}$$

$$-a_{121}\cdot a_{212}\cdot a_{333} + a_{121}\cdot a_{213}\cdot a_{332} + a_{121}\cdot a_{232}\cdot a_{313} - a_{121}\cdot a_{233}\cdot a_{312}$$

$$+a_{122}\cdot a_{211}\cdot a_{333} - a_{122}\cdot a_{213}\cdot a_{331} - a_{122}\cdot a_{231}\cdot a_{313} + a_{122}\cdot a_{233}\cdot a_{311}$$

$$-a_{123}\cdot a_{211}\cdot a_{332} + a_{123}\cdot a_{212}\cdot a_{331} + a_{123}\cdot a_{231}\cdot a_{312} - a_{123}\cdot a_{232}\cdot a_{311}$$

$$+a_{131}\cdot a_{212}\cdot a_{323}-a_{131}\cdot a_{213}\cdot a_{322}-a_{131}\cdot a_{222}\cdot a_{313}+a_{131}\cdot a_{223}\cdot a_{312}$$

$$-a_{132}\cdot a_{211}\cdot a_{323}+a_{132}\cdot a_{213}\cdot a_{321}+a_{132}\cdot a_{221}\cdot a_{313}-a_{132}\cdot a_{223}\cdot a_{311}$$

$$+a_{133}\cdot a_{211}\cdot a_{322}-a_{133}\cdot a_{212}\cdot a_{321}-a_{133}\cdot a_{221}\cdot a_{312}+a_{133}\cdot a_{222}\cdot a_{311})=-\det[A_{3\times3\times3}]$$

2. For interchanging the location of two consecutive "vertical page": Let B be cubic-matrix of order 3, where we have interchanged two consecutive "vertical page" (Second page with third page), then based on definition 3:

$$\det[B_{3\times3\times3}]=\det\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{131} & a_{121} & a_{112} & a_{132} & a_{122} & a_{113} & a_{133} & a_{123}\\ a_{211} & a_{231} & a_{221} & a_{212} & a_{232} & a_{222} & a_{213} & a_{233} & a_{223}\\ a_{311} & a_{331} & a_{321} & a_{312} & a_{332} & a_{322} & a_{313} & a_{333} & a_{323}\end{array}\right)$$

$$=a_{111}\cdot a_{232}\cdot a_{323}-a_{111}\cdot a_{222}\cdot a_{333}-a_{111}\cdot a_{233}\cdot a_{322}+a_{111}\cdot a_{223}\cdot a_{332}$$

$$-a_{112}\cdot a_{231}\cdot a_{323}+a_{112}\cdot a_{233}\cdot a_{321}+a_{112}\cdot a_{221}\cdot a_{333}-a_{112}\cdot a_{223}\cdot a_{331}$$

$$+a_{113}\cdot a_{231}\cdot a_{322}-a_{113}\cdot a_{232}\cdot a_{321}-a_{113}\cdot a_{221}\cdot a_{332}+a_{113}\cdot a_{222}\cdot a_{331}$$

$$-a_{131}\cdot a_{212}\cdot a_{323}+a_{131}\cdot a_{213}\cdot a_{322}+a_{131}\cdot a_{222}\cdot a_{313}-a_{131}\cdot a_{223}\cdot a_{312}$$

$$+a_{132}\cdot a_{211}\cdot a_{323}-a_{132}\cdot a_{213}\cdot a_{321}-a_{132}\cdot a_{221}\cdot a_{313}+a_{132}\cdot a_{223}\cdot a_{311}$$

$$-a_{133}\cdot a_{211}\cdot a_{322}+a_{133}\cdot a_{212}\cdot a_{321}+a_{133}\cdot a_{221}\cdot a_{312}-a_{133}\cdot a_{222}\cdot a_{311}$$

$$+a_{121}\cdot a_{212}\cdot a_{333}-a_{121}\cdot a_{213}\cdot a_{332}-a_{121}\cdot a_{232}\cdot a_{313}+a_{121}\cdot a_{233}\cdot a_{312}$$

$$-a_{122}\cdot a_{211}\cdot a_{333}+a_{122}\cdot a_{213}\cdot a_{331}+a_{122}\cdot a_{231}\cdot a_{313}-a_{122}\cdot a_{233}\cdot a_{311}$$

$$+a_{123}\cdot a_{211}\cdot a_{332}-a_{123}\cdot a_{212}\cdot a_{331}-a_{123}\cdot a_{231}\cdot a_{312}+a_{123}\cdot a_{232}\cdot a_{311}$$

$$=-a_{111}\cdot a_{222}\cdot a_{333}+a_{111}\cdot a_{232}\cdot a_{323}+a_{111}\cdot a_{223}\cdot a_{332}-a_{111}\cdot a_{233}\cdot a_{322}$$

$$+a_{112}\cdot a_{221}\cdot a_{333}-a_{112}\cdot a_{223}\cdot a_{331}-a_{112}\cdot a_{231}\cdot a_{323}+a_{112}\cdot a_{233}\cdot a_{321}$$

$$-a_{113}\cdot a_{221}\cdot a_{332}+a_{113}\cdot a_{222}\cdot a_{331}+a_{113}\cdot a_{231}\cdot a_{322}-a_{113}\cdot a_{232}\cdot a_{321}$$

$$+a_{121}\cdot a_{212}\cdot a_{333}-a_{121}\cdot a_{213}\cdot a_{332}-a_{121}\cdot a_{232}\cdot a_{313}+a_{121}\cdot a_{233}\cdot a_{312}$$

$$-a_{122}\cdot a_{211}\cdot a_{333}+a_{122}\cdot a_{213}\cdot a_{331}+a_{122}\cdot a_{231}\cdot a_{313}-a_{122}\cdot a_{233}\cdot a_{311}$$

$$+a_{123}\cdot a_{211}\cdot a_{332}-a_{123}\cdot a_{212}\cdot a_{331}-a_{123}\cdot a_{231}\cdot a_{312}+a_{123}\cdot a_{232}\cdot a_{311}$$

$$-a_{131}\cdot a_{212}\cdot a_{323}+a_{131}\cdot a_{213}\cdot a_{322}+a_{131}\cdot a_{222}\cdot a_{313}-a_{131}\cdot a_{223}\cdot a_{312}$$

$$+a_{132}\cdot a_{211}\cdot a_{323}-a_{132}\cdot a_{213}\cdot a_{321}-a_{132}\cdot a_{221}\cdot a_{313}+a_{132}\cdot a_{223}\cdot a_{311}$$

$$-a_{133}\cdot a_{211}\cdot a_{322}+a_{133}\cdot a_{212}\cdot a_{321}+a_{133}\cdot a_{221}\cdot a_{312}-a_{133}\cdot a_{222}\cdot a_{311}$$

$$=-(a_{111}\cdot a_{222}\cdot a_{333}-a_{111}\cdot a_{232}\cdot a_{323}-a_{111}\cdot a_{223}\cdot a_{332}+a_{111}\cdot a_{233}\cdot a_{322}$$

$$-a_{112}\cdot a_{221}\cdot a_{333}+a_{112}\cdot a_{223}\cdot a_{331}+a_{112}\cdot a_{231}\cdot a_{323}-a_{112}\cdot a_{233}\cdot a_{321}$$

$$+a_{113}\cdot a_{221}\cdot a_{332}-a_{113}\cdot a_{222}\cdot a_{331}-a_{113}\cdot a_{231}\cdot a_{322}+a_{113}\cdot a_{232}\cdot a_{321}$$

$$-a_{121}\cdot a_{212}\cdot a_{333}+a_{121}\cdot a_{213}\cdot a_{332}+a_{121}\cdot a_{232}\cdot a_{313}-a_{121}\cdot a_{233}\cdot a_{312}$$
$$+a_{122}\cdot a_{211}\cdot a_{333}-a_{122}\cdot a_{213}\cdot a_{331}-a_{122}\cdot a_{231}\cdot a_{313}+a_{122}\cdot a_{233}\cdot a_{311}$$
$$-a_{123}\cdot a_{211}\cdot a_{332}+a_{123}\cdot a_{212}\cdot a_{331}+a_{123}\cdot a_{231}\cdot a_{312}-a_{123}\cdot a_{232}\cdot a_{311}$$
$$+a_{131}\cdot a_{212}\cdot a_{323}-a_{131}\cdot a_{213}\cdot a_{322}-a_{131}\cdot a_{222}\cdot a_{313}+a_{131}\cdot a_{223}\cdot a_{312}$$
$$-a_{132}\cdot a_{211}\cdot a_{323}+a_{132}\cdot a_{213}\cdot a_{321}+a_{132}\cdot a_{221}\cdot a_{313}-a_{132}\cdot a_{223}\cdot a_{311}$$
$$+a_{133}\cdot a_{211}\cdot a_{322}-a_{133}\cdot a_{212}\cdot a_{321}-a_{133}\cdot a_{221}\cdot a_{312}+a_{133}\cdot a_{222}\cdot a_{311}) = -\det[A_{3\times3\times3}]$$

3. For interchanging the location of two consecutive "vertical layers": Let B be cubic-matrix of order 3, where we have interchanged 2-consecutive "vertical layers" (First layer with second layer), then based on definition 3:

$$\det[B_{3\times3\times3}] = \det\left(\begin{array}{ccc|ccc|ccc} a_{112} & a_{122} & a_{132} & a_{111} & a_{121} & a_{131} & a_{113} & a_{123} & a_{133} \\ a_{212} & a_{222} & a_{232} & a_{211} & a_{221} & a_{231} & a_{213} & a_{223} & a_{233} \\ a_{312} & a_{322} & a_{332} & a_{311} & a_{321} & a_{331} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$= a_{112}\cdot a_{221}\cdot a_{333}-a_{112}\cdot a_{231}\cdot a_{323}-a_{112}\cdot a_{223}\cdot a_{331}+a_{112}\cdot a_{233}\cdot a_{321}$$
$$-a_{111}\cdot a_{222}\cdot a_{333}+a_{111}\cdot a_{223}\cdot a_{332}+a_{111}\cdot a_{232}\cdot a_{323}-a_{111}\cdot a_{233}\cdot a_{322}$$
$$+a_{113}\cdot a_{222}\cdot a_{331}-a_{113}\cdot a_{221}\cdot a_{332}-a_{113}\cdot a_{232}\cdot a_{321}+a_{113}\cdot a_{231}\cdot a_{322}$$
$$-a_{122}\cdot a_{211}\cdot a_{333}+a_{122}\cdot a_{213}\cdot a_{331}+a_{122}\cdot a_{231}\cdot a_{313}-a_{122}\cdot a_{233}\cdot a_{311}$$
$$+a_{121}\cdot a_{212}\cdot a_{333}-a_{121}\cdot a_{213}\cdot a_{332}-a_{121}\cdot a_{232}\cdot a_{313}+a_{121}\cdot a_{233}\cdot a_{312}$$
$$-a_{123}\cdot a_{212}\cdot a_{331}+a_{123}\cdot a_{211}\cdot a_{332}+a_{123}\cdot a_{232}\cdot a_{311}-a_{123}\cdot a_{231}\cdot a_{312}$$
$$+a_{132}\cdot a_{211}\cdot a_{323}-a_{132}\cdot a_{213}\cdot a_{321}-a_{132}\cdot a_{221}\cdot a_{313}+a_{132}\cdot a_{223}\cdot a_{311}$$
$$-a_{131}\cdot a_{212}\cdot a_{323}+a_{131}\cdot a_{213}\cdot a_{322}+a_{131}\cdot a_{222}\cdot a_{313}-a_{131}\cdot a_{223}\cdot a_{312}$$
$$+a_{133}\cdot a_{212}\cdot a_{321}-a_{133}\cdot a_{211}\cdot a_{322}-a_{133}\cdot a_{222}\cdot a_{311}+a_{133}\cdot a_{221}\cdot a_{312}$$
$$= -a_{111}\cdot a_{222}\cdot a_{333}+a_{111}\cdot a_{232}\cdot a_{323}+a_{111}\cdot a_{223}\cdot a_{332}-a_{111}\cdot a_{233}\cdot a_{322}$$
$$+a_{112}\cdot a_{221}\cdot a_{333}-a_{112}\cdot a_{223}\cdot a_{331}-a_{112}\cdot a_{231}\cdot a_{323}+a_{112}\cdot a_{233}\cdot a_{321}$$
$$-a_{113}\cdot a_{221}\cdot a_{332}+a_{113}\cdot a_{222}\cdot a_{331}+a_{113}\cdot a_{231}\cdot a_{322}-a_{113}\cdot a_{232}\cdot a_{321}$$
$$+a_{121}\cdot a_{212}\cdot a_{333}-a_{121}\cdot a_{213}\cdot a_{332}-a_{121}\cdot a_{232}\cdot a_{313}+a_{121}\cdot a_{233}\cdot a_{312}$$
$$-a_{122}\cdot a_{211}\cdot a_{333}+a_{122}\cdot a_{213}\cdot a_{331}+a_{122}\cdot a_{231}\cdot a_{313}-a_{122}\cdot a_{233}\cdot a_{311}$$
$$+a_{123}\cdot a_{211}\cdot a_{332}-a_{123}\cdot a_{212}\cdot a_{331}-a_{123}\cdot a_{231}\cdot a_{312}+a_{123}\cdot a_{232}\cdot a_{311}$$
$$-a_{131}\cdot a_{212}\cdot a_{323}+a_{131}\cdot a_{213}\cdot a_{322}+a_{131}\cdot a_{222}\cdot a_{313}-a_{131}\cdot a_{223}\cdot a_{312}$$
$$+a_{132}\cdot a_{211}\cdot a_{323}-a_{132}\cdot a_{213}\cdot a_{321}-a_{132}\cdot a_{221}\cdot a_{313}+a_{132}\cdot a_{223}\cdot a_{311}$$
$$-a_{133}\cdot a_{211}\cdot a_{322}+a_{133}\cdot a_{212}\cdot a_{321}+a_{133}\cdot a_{221}\cdot a_{312}-a_{133}\cdot a_{222}\cdot a_{311}$$

$$= -(a_{111} \cdot a_{222} \cdot a_{333} - a_{111} \cdot a_{232} \cdot a_{323} - a_{111} \cdot a_{223} \cdot a_{332} + a_{111} \cdot a_{233} \cdot a_{322}$$

$$-a_{112} \cdot a_{221} \cdot a_{333} + a_{112} \cdot a_{223} \cdot a_{331} + a_{112} \cdot a_{231} \cdot a_{323} - a_{112} \cdot a_{233} \cdot a_{321}$$

$$+a_{113} \cdot a_{221} \cdot a_{332} - a_{113} \cdot a_{222} \cdot a_{331} - a_{113} \cdot a_{231} \cdot a_{322} + a_{113} \cdot a_{232} \cdot a_{321}$$

$$-a_{121} \cdot a_{212} \cdot a_{333} + a_{121} \cdot a_{213} \cdot a_{332} + a_{121} \cdot a_{232} \cdot a_{313} - a_{121} \cdot a_{233} \cdot a_{312}$$

$$+a_{122} \cdot a_{211} \cdot a_{333} - a_{122} \cdot a_{213} \cdot a_{331} - a_{122} \cdot a_{231} \cdot a_{313} + a_{122} \cdot a_{233} \cdot a_{311}$$

$$-a_{123} \cdot a_{211} \cdot a_{332} + a_{123} \cdot a_{212} \cdot a_{331} + a_{123} \cdot a_{231} \cdot a_{312} - a_{123} \cdot a_{232} \cdot a_{311}$$

$$+a_{131} \cdot a_{212} \cdot a_{323} - a_{131} \cdot a_{213} \cdot a_{322} - a_{131} \cdot a_{222} \cdot a_{313} + a_{131} \cdot a_{223} \cdot a_{312}$$

$$-a_{132} \cdot a_{211} \cdot a_{323} + a_{132} \cdot a_{213} \cdot a_{321} + a_{132} \cdot a_{221} \cdot a_{313} - a_{132} \cdot a_{223} \cdot a_{311}$$

$$+a_{133} \cdot a_{211} \cdot a_{322} - a_{133} \cdot a_{212} \cdot a_{321} - a_{133} \cdot a_{221} \cdot a_{312} + a_{133} \cdot a_{222} \cdot a_{311}) = -\det[A_{3\times 3\times 3}]$$

4. For interchanging the location of two consecutive "vertical layers": Let B be cubic-matrix of order 3, where we have interchanged 2-consecutive "vertical layers" (Second layer with third layer), then based on definition 3:

$$\det[B_{3\times 3\times 3}] = \det\left(\begin{matrix} a_{111} & a_{121} & a_{131} \\ a_{211} & a_{221} & a_{231} \\ a_{311} & a_{321} & a_{331} \end{matrix}\middle| \begin{matrix} a_{113} & a_{123} & a_{133} \\ a_{213} & a_{223} & a_{233} \\ a_{313} & a_{323} & a_{333} \end{matrix}\middle| \begin{matrix} a_{112} & a_{122} & a_{132} \\ a_{212} & a_{222} & a_{232} \\ a_{312} & a_{322} & a_{332} \end{matrix}\right)$$

$$= a_{111} \cdot a_{223} \cdot a_{332} - a_{111} \cdot a_{233} \cdot a_{322} - a_{111} \cdot a_{222} \cdot a_{333} + a_{111} \cdot a_{232} \cdot a_{323}$$

$$-a_{113} \cdot a_{221} \cdot a_{332} + a_{113} \cdot a_{222} \cdot a_{331} + a_{113} \cdot a_{231} \cdot a_{322} - a_{113} \cdot a_{232} \cdot a_{321}$$

$$+a_{112} \cdot a_{221} \cdot a_{333} - a_{112} \cdot a_{223} \cdot a_{331} - a_{112} \cdot a_{231} \cdot a_{323} + a_{112} \cdot a_{233} \cdot a_{321}$$

$$-a_{121} \cdot a_{213} \cdot a_{332} + a_{121} \cdot a_{212} \cdot a_{333} + a_{121} \cdot a_{233} \cdot a_{312} - a_{121} \cdot a_{232} \cdot a_{313}$$

$$+a_{123} \cdot a_{211} \cdot a_{332} - a_{123} \cdot a_{212} \cdot a_{331} - a_{123} \cdot a_{231} \cdot a_{312} + a_{123} \cdot a_{232} \cdot a_{311}$$

$$-a_{122} \cdot a_{211} \cdot a_{333} + a_{122} \cdot a_{213} \cdot a_{331} + a_{122} \cdot a_{231} \cdot a_{313} - a_{122} \cdot a_{233} \cdot a_{311}$$

$$+a_{131} \cdot a_{213} \cdot a_{322} - a_{131} \cdot a_{212} \cdot a_{323} - a_{131} \cdot a_{223} \cdot a_{312} + a_{131} \cdot a_{222} \cdot a_{313}$$

$$-a_{133} \cdot a_{211} \cdot a_{322} + a_{133} \cdot a_{212} \cdot a_{321} + a_{133} \cdot a_{221} \cdot a_{312} - a_{133} \cdot a_{222} \cdot a_{311}$$

$$+a_{132} \cdot a_{211} \cdot a_{323} - a_{132} \cdot a_{213} \cdot a_{321} - a_{132} \cdot a_{221} \cdot a_{313} + a_{132} \cdot a_{223} \cdot a_{311}$$

$$= -a_{111} \cdot a_{222} \cdot a_{333} + a_{111} \cdot a_{232} \cdot a_{323} + a_{111} \cdot a_{223} \cdot a_{332} - a_{111} \cdot a_{233} \cdot a_{322}$$

$$+a_{112} \cdot a_{221} \cdot a_{333} - a_{112} \cdot a_{223} \cdot a_{331} - a_{112} \cdot a_{231} \cdot a_{323} + a_{112} \cdot a_{233} \cdot a_{321}$$

$$-a_{113} \cdot a_{221} \cdot a_{332} + a_{113} \cdot a_{222} \cdot a_{331} + a_{113} \cdot a_{231} \cdot a_{322} - a_{113} \cdot a_{232} \cdot a_{321}$$

$$+a_{121} \cdot a_{212} \cdot a_{333} - a_{121} \cdot a_{213} \cdot a_{332} - a_{121} \cdot a_{232} \cdot a_{313} + a_{121} \cdot a_{233} \cdot a_{312}$$

$$-a_{122} \cdot a_{211} \cdot a_{333} + a_{122} \cdot a_{213} \cdot a_{331} + a_{122} \cdot a_{231} \cdot a_{313} - a_{122} \cdot a_{233} \cdot a_{311}$$

$$+a_{123} \cdot a_{211} \cdot a_{332} - a_{123} \cdot a_{212} \cdot a_{331} - a_{123} \cdot a_{231} \cdot a_{312} + a_{123} \cdot a_{232} \cdot a_{311}$$

$$-a_{131}\cdot a_{212}\cdot a_{323}+a_{131}\cdot a_{213}\cdot a_{322}+a_{131}\cdot a_{222}\cdot a_{313}-a_{131}\cdot a_{223}\cdot a_{312}$$

$$+a_{132}\cdot a_{211}\cdot a_{323}-a_{132}\cdot a_{213}\cdot a_{321}-a_{132}\cdot a_{221}\cdot a_{313}+a_{132}\cdot a_{223}\cdot a_{311}$$

$$-a_{133}\cdot a_{211}\cdot a_{322}+a_{133}\cdot a_{212}\cdot a_{321}+a_{133}\cdot a_{221}\cdot a_{312}-a_{133}\cdot a_{222}\cdot a_{311}$$

$$=-(a_{111}\cdot a_{222}\cdot a_{333}-a_{111}\cdot a_{232}\cdot a_{323}-a_{111}\cdot a_{223}\cdot a_{332}+a_{111}\cdot a_{233}\cdot a_{322}$$

$$-a_{112}\cdot a_{221}\cdot a_{333}+a_{112}\cdot a_{223}\cdot a_{331}+a_{112}\cdot a_{231}\cdot a_{323}-a_{112}\cdot a_{233}\cdot a_{321}$$

$$+a_{113}\cdot a_{221}\cdot a_{332}-a_{113}\cdot a_{222}\cdot a_{331}-a_{113}\cdot a_{231}\cdot a_{322}+a_{113}\cdot a_{232}\cdot a_{321}$$

$$-a_{121}\cdot a_{212}\cdot a_{333}+a_{121}\cdot a_{213}\cdot a_{332}+a_{121}\cdot a_{232}\cdot a_{313}-a_{121}\cdot a_{233}\cdot a_{312}$$

$$+a_{122}\cdot a_{211}\cdot a_{333}-a_{122}\cdot a_{213}\cdot a_{331}-a_{122}\cdot a_{231}\cdot a_{313}+a_{122}\cdot a_{233}\cdot a_{311}$$

$$-a_{123}\cdot a_{211}\cdot a_{332}+a_{123}\cdot a_{212}\cdot a_{331}+a_{123}\cdot a_{231}\cdot a_{312}-a_{123}\cdot a_{232}\cdot a_{311}$$

$$+a_{131}\cdot a_{212}\cdot a_{323}-a_{131}\cdot a_{213}\cdot a_{322}-a_{131}\cdot a_{222}\cdot a_{313}+a_{131}\cdot a_{223}\cdot a_{312}$$

$$-a_{132}\cdot a_{211}\cdot a_{323}+a_{132}\cdot a_{213}\cdot a_{321}+a_{132}\cdot a_{221}\cdot a_{313}-a_{132}\cdot a_{223}\cdot a_{311}$$

$$+a_{133}\cdot a_{211}\cdot a_{322}-a_{133}\cdot a_{212}\cdot a_{321}-a_{133}\cdot a_{221}\cdot a_{312}+a_{133}\cdot a_{222}\cdot a_{311})=-\det[A_{3\times3\times3}]$$

Example 6 *Let A be $2\times2\times2$ 3D determinant than we will obtain determinant B from A by interchanging location of two plans in j index:*

$$\det[A_{2\times2\times2}]=\det\begin{pmatrix}2 & 1 & 4 & 7\\ 3 & 5 & 3 & 2\end{pmatrix}=2\cdot2-4\cdot5-1\cdot3+7\cdot3=2.$$

Then:

$$\det[B_{2\times2\times2}]=\det\begin{pmatrix}1 & 2 & 7 & 4\\ 5 & 3 & 2 & 3\end{pmatrix}=1\cdot3-7\cdot3-2\cdot2+4\cdot5=-2.$$

Example 7 *Let A be a $3\times3\times3$ 3D determinant than we will obtain determinant B from A by interchanging location of two plans in j index:*

$$\det[A_{3\times3\times3}]=\det\begin{pmatrix}1 & 4 & 2 & 3 & 1 & 3 & 2 & 1 & 0\\ 2 & 0 & 0 & 5 & 1 & 3 & 0 & 1 & 0\\ 0 & 4 & 2 & 3 & 2 & 0 & 2 & 1 & 0\end{pmatrix}=-3+6-4+24+24+10+10-4+6-6=63$$

Then:

$$\det[A_{3\times3\times3}]=\det\begin{pmatrix}4 & 1 & 2 & 1 & 3 & 3 & 1 & 2 & 0\\ 0 & 2 & 0 & 1 & 5 & 3 & 1 & 0 & 0\\ 4 & 0 & 2 & 2 & 3 & 0 & 1 & 2 & 0\end{pmatrix}=-24-10-6+3-6-24+4+4+6-10=-63$$

If we compare results with example 2, we can see that $|A|=-|B|$

Example 8 *Let A be $2\times2\times2$ 3D determinant than we will obtain determinant B from A by interchanging location of two plans in k index:*

$$\det[A_{2\times2\times2}]=\det\begin{pmatrix}2 & 1 & 4 & 7\\ 3 & 5 & 3 & 2\end{pmatrix}=2\cdot2-4\cdot5-1\cdot3+7\cdot3=2.$$

Then:

$$\det[B_{2\times2\times2}] = \det\begin{pmatrix} 4 & 7 & 2 & 1 \\ 3 & 2 & 3 & 5 \end{pmatrix} = 4\cdot5-2\cdot2-7\cdot3+1\cdot3=-2.$$

If we compare results with example 2, we can see that $|A|=-|B|$.

Example 9 *Let A be a $3\times3\times3$ 3D determinant than we will obtain determinant B from A by interchanging location of two plans in k index:*

$$\det[A_{3\times3\times3}] = \det\begin{pmatrix} 1 & 4 & 2 & 3 & 1 & 3 & 2 & 1 & 0 \\ 2 & 0 & 0 & 5 & 1 & 3 & 0 & 1 & 0 \\ 0 & 4 & 2 & 3 & 2 & 0 & 2 & 1 & 0 \end{pmatrix} = -3+6-4+24+24+10+10-4+6-6=63$$

Then:

$$\det[B_{3\times3\times3}] = \det\begin{pmatrix} 3 & 1 & 3 & 1 & 4 & 2 & 2 & 1 & 0 \\ 5 & 1 & 3 & 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 4 & 2 & 2 & 1 & 0 \end{pmatrix} = -6+3-24-24-10-6+6-10-4-4=-63$$

If we compare results with example 3, we can see that $|A|=-|B|$.

Algorithms Implementation of Determinants for Cubic-Matrix of Order 2 and 3

Computer Algorithm for Calculating Determinant of Cubic Matrices of Order 2 and Order 3

In the following we have presented the pseudocode of algorithm for calculating the determinant of cubic matrices of order 2 and order 3based on the Definition 2 and Definition 3.

tw]

P 1: Determinant calculation of cubic matrices of second and third order

tw]

Step 1: Determine the order of determinant:
$[m, n, o] = size(A);$

Step 2: Checking if 3D matrix is cubic:
if $m \sim n; m \sim= o; n \sim= o;$
disp('A is not square, cannot calculate the determinant')
$d = 0;$
Return
end

Step 3: Checking if 3D matrix is higher than the 3rd order:
if $m > 3;$
disp('A is higher than the third order, hence can not be calculated.')
$d = 0;$
return
end

Step 4: Initialize $d = 0;$

Step 5: Handling base case.
if $m == 1$
$d = A;$

return
end

Step 6: Check if A is of second order.
$d = A(1,1,1) * A(2,2,2) - A(1,1,2) * A(2,2,1) - A(1,2,1) * A(2,1,2) + A(1,2,2) * A(2,1,1)$

Step 7: Check if A is of third order.
$d =$

$$
\begin{aligned}
&= A(1,1,1) * A(2,2,2) * A(3,3,3) - A(1,1,1) * A(2,3,2) * A(3,2,3) - A(1,1,1) * A(2,2,3) * A(3,3,2) \\
&+A(1,1,1) * A(2,3,3) * A(3,2,2) - A(1,1,2) * A(2,2,1) * A(3,3,3) + A(1,1,2) * A(2,2,3) * A(3,3,1) \\
&+A(1,1,2) * A(2,3,1) * A(3,2,3) - A(1,1,2) * A(2,3,3) * A(3,2,1) + A(1,1,3) * A(2,2,1) * A(3,3,2) \\
&-A(1,1,3) * A(2,2,2) * A(3,3,1) - A(1,1,3) * A(2,3,1) * A(3,2,2) + A(1,1,3) * A(2,3,2) * A(3,2,1) \\
&-A(1,2,1) * A(2,1,2) * A(3,3,3) + A(1,2,1) * A(2,1,3) * A(3,3,2) + A(1,2,1) * A(2,3,2) * A(3,1,3) \\
&-A(1,2,1) * A(2,3,3) * A(3,1,2) + A(1,2,2) * A(2,1,1) * A(3,3,3) - A(1,2,2) * A(2,1,3) * A(3,3,1) \\
&-A(1,2,2) * A(2,3,1) * A(3,1,3) + A(1,2,2) * A(2,3,3) * A(3,1,1) - A(1,2,3) * A(2,1,1) * A(3,3,2) \\
&+A(1,2,3) * A(2,1,2) * A(3,3,1) + A(1,2,3) * A(2,3,1) * A(3,1,2) - A(1,2,3) * A(2,3,2) * A(3,1,1) \\
&+A(1,3,1) * A(2,1,2) * A(3,2,3) - A(1,3,1) * A(2,1,3) * A(3,2,2) - A(1,3,1) * A(2,2,2) * A(3,1,3) \\
&+A(1,3,1) * A(2,2,3) * A(3,1,2) - A(1,3,2) * A(2,1,1) * A(3,2,3) + A(1,3,2) * A(2,1,3) * A(3,2,1) \\
&+A(1,3,2) * A(2,2,1) * A(3,1,3) - A(1,3,2) * A(2,2,3) * A(3,1,1) + A(1,3,3) * A(2,1,1) * A(3,2,2) \\
&-A(1,3,3) * A(2,1,2) * A(3,2,1) - A(1,3,3) * A(2,2,1) * A(3,1,2) + A(1,3,3) * A(2,2,2) * A(3,1,1)
\end{aligned}
$$

Step 8: Return the result of 3D determinant.
tw]

Optimized Version of Computer Algorithm

The above-mention algorithm is hard-coded exactly as it is prescribed in the Definition 2 and Definition 3, it can be optimized further with nested-loop. Hence, in the following we have presented another optimized version of above algorithm which gives the same result.

tw]

P 2: Optimized algorithm for determinant calculation of cubic matrices of second and third order

tw]

Step 1: Determine the order of determinant:
$[m, n, o] = size(A);$

Step 2: Checking if 3D matrix is cubic:
if $m \sim n; m \sim= o; n \sim= o;$
disp('A is not square, cannot calculate the determinant')
$d = 0;$
return

end

Step 3: Checking if 3D matrix is higher than the 3rd order:
if $m > 3$;
disp('A is higher than the third order, hence can not be calculated.')
$d = 0$;
return
end

Step 4: Initialize $d = 0$;

Step 5: Handling base case.
if $m == 1$
$d = A$;
return
end

Step 6: Check if A is of second order.
Create loop for j from 1 to 2
Create loop for k from 1 to 2
Calculate determinant:

$$d = d + (-1)^\wedge(2 + j + k) * A(1, j, k) * det\_3D(A(2, [1{:}j - 1 j + 1{:}2], [1{:}k - 1 k + 1{:}2]));$$

end
end

Step 7: Check if A is of third order.
Create loop for j from 1 to 3
Create loop for k from 1 to 3
Calculate determinant:

$$d = d + (-1)^\wedge(2 + j + k) * A(1, j, k) * det\_3D(A(2{:}3, [1{:}j - 1 j + 1{:}3], [1{:}k - 1 k + 1{:}3]));$$

end
end

Step 8: Return the result of 3D determinant.
tw]

Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

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