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On Mersenne GCED Matrices

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Abstract: A Mersenne number is defined as a number of the form $M_n = 2^n - 1$, where *n* is a positive integer. The first five Mersenne numbers are 1, 3, 7, 15, and 31. A divisor *d* of a positive integer $m = p^k$, where *p* is a prime, is termed an exponential divisor if it satisfies $d = p^t$ with *t* dividing *k*, and it is denoted as $d|_e m$. Two integers *a* and *b* share a common exponential divisor if they have the same prime factors. The greatest common exponential divisor (*GCED*) of two integers *a* and *b* is denoted by *gced*(*a, b*). A set *S* is called exponential factor-closed if the exponential divisor of every element of *S* also belongs to *S*. Similarly, *S* is *GCED*-closed if *gced*(*a, b*) belongs to *S* for every pair a, b in *S*. If *S* is an exponential factor-closed set of distinct positive integers arranged in increasing order, the *GCED* matrix associated with *S* is the matrix *M*, where each entry M_{ij} is given by *gced*(a_i, a_j). The Mersenne *GCED* matrix *M* associated with *S* is a square matrix where each entry M_{ij} is of the form *gced*$(2^{a_i} - 1, 2^{a_j} - 1)$. This paper introduces the concept of Mersenne GCED square matrices defined on a non-exponential factor-closed set. We establish a comprehensive characterization of their fundamental properties, including their structure, determinant, reciprocal, and inverse.

Keywords: Exponential divisor, Greatest common divisor, Mersenne numbers, Factor-closed set, GCED-closed set.

Introduction

Let $S = \{x_1, x_2, \dots, x_n\}$ be a well-ordered set of distinct positive integers. The $n \times n$ matrix $A = a_{ij}$, $(\text{resp.} [A] = a_{ij})$ where $a_{ij} = (x_i, x_j)$ $(\text{resp.} a_{ij} = [x_i, x_j])$ represents the greatest common divisor of x_i and x_j (resp. the least common multiple of x_i and x_j), which is denoted by GCD (resp. LCM) matrix on the set S. The set S is a factor closed set if it includes every divisor of any $x \in S$ while S is a GCD closed set if it contains (x_i, x_j) for all x_i and x_j in S.

Mersenne numbers are a sequence of numbers of the form $M_n = 2^n - 1$, where *n* is a positive integer. The first five Mersenne numbers are 1, 3, 7, 15, and 31. If M_n is a prime number, it is called a Mersenne prime. It is a necessary (but not sufficient) condition: For M_n to be prime, n itself must be prime. However, not all numbers M_p (where p is prime) are Mersenne primes. For example, $M_{11} = 2047 = 23 \times 89$ is composite. Mersenne primes hold great significance in number theory, particularly due to their close relationship with perfect numbers (numbers that are equal to the sum of their proper divisors, excluding themselves), as established by the Euclid-Euler theorem. They also play a crucial role in primality testing, with the Lucas-Lehmer test being specifically designed to determine the primality of Mersenne numbers efficiently, see Awad et al. (2023a) for more details.

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Moreover, the largest known primes are often Mersenne primes, as their structure allows for more effective primality-checking algorithms. Despite their importance, an open problem in mathematics remains unresolved: it is still unknown whether there exist infinitely many Mersenne primes. Subrarao (1972) introduced the concept of exponential divisors, where $d = \prod_{i=1}^{t} p_i^{a_i}$ is an exponential divisor of $m = \prod_{i=1}^{t} p_i^{b_i}$ if a_i divides b_i for all $1 \leq i \leq t$ denoted by $d \mid_e m$. By convention, $1 \mid_e 1$ but 1 is not an exponential divisor for every $m > 1$. If two integers m and n share the same prime factors, they have a common exponential divisor. The greatest common exponential divisor (GCED) and the least common exponential multiple (LCEM) of m and n are denoted by $(m, n)_e$ and $[m, n]_e$ respectively. By convention, $(1,1)_e = [1,1]_e = 1$ and $(1, m)_e$ and $[1, m]_e$ do not exist for $m > 1$. If m and n have the same prime factorization $m = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k}$ and $n = p_1^{c_1} p_2^{c_2} \dots p_k^{c_k}$, then:

$$(m, n)_e = \prod_{i=1}^{k} p_i^{gcd(b_i, c_i)}.$$

Two integers $m = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k}$ and $n = p_1^{c_1} p_2^{c_2} \dots p_k^{c_k}$ are exponentially relatively prime if $(m, n)_e = 1$ or if $\gcd(b_i, c_i) = 1$ for all $1 \leq i \leq k$.

A set $S = \{x_1, x_2, \dots, x_n\}$ is exponential factor-closed (or GCED-closed) if it contains all exponential divisors of its elements (or if $(x_i, x_j)_e \in S$ for all $x_i, x_j \in S$). For example the set $\{20, 50, 100\}$ is not exponential factor closed set where as $\{10, 20, 50, 100\}$ is. The latter is also $GCED$-closed. If S is an exponential factor-closed set arranged in increasing order, the $n \times n$ matrix $(S)_e = s_{ij}$ where $s_{ij} = (x_i, x_j)_e$, is called the GCED matrix on S.

Smith (1875) showed that if $T = \{1,2, \dots, n\}$, then

$$det(T) = \prod_{i=1}^{n} \phi(i), \text{ and } det[T] = \prod_{i=1}^{n} \phi(x_i)\pi(x_i),$$

where ϕ is Euler's totient function and π is a multiplicative function such that $\pi(p^k) = -p$ for a prime p. Smith also extended these results to factor-closed sets. Beslin and Ligh (1989b), later factorized GCD matrices proving their non-singularity. In subsequent works Beslin (1989a, 1992) they further analyzed GCD matrices over GCD-closed sets, computing their determinants. Since Smith's foundational work, this field has expanded significantly, with numerous studies generalizing the structure, determinants, and inverses of GCD and LCM matrices. The study gained momentum in 1989, particularly due to Beslin and Ligh's contributions. They established that for a factor-closed set S, the GCD matrix S can be decomposed as AA^T, where A is upper triangular and further proved that GCD matrices are positive definite. For a well-ordered set $S = \{x_1, x_2, \dots, x_n\}$ with $x_1 < x_2 < \dots < x_n$, the $n \times n$ power gcd matrix $(M^r) = \gcd(x_i, x_j)^r$, where r is any real number.

The set $(\mathbf{Z}^+ \backslash \{1\}, |_e)$ under exponential divisibility forms a poset but not a lattice, as the GCED does not always exist. Raza and Waheed (2015a, 2015b, 2012) derived structure theorems and computed determinants for GCED and LCEM matrices on ordered sets. Zeid et al. (2022) studied GCED matrices defined on both GCED-closed and non-GCED-closed sets over a unique factorization domain D. Additionally, they provided a comprehensive characterization of their structure, determinant, trace, and inverse. Chehade et al. (2024) investigated LCEM matrices over UFDs, providing a complete characterization of their structure, determinant, trace, and inverse. Awad et al. (2019) introduced the Fermat power GCD matrices defined on sets that were neither factor-closed nor GCD-closed. They establishedd a complete characterization for their matrix factorizations, determinants, reciprocal forms, and inverses. Awad et al. (2020) employed a generalized Jordan totient function to extend the theory of reciprocal power GCDQ and LCMQ matrices from the classical setting of natural integers to Euclidean domains. Their results included structural theorems and determinantal formulas for these matrices, which were applicable to both arbitrary and factor-closed q-ordered sets within such domains. Awad et al. (2023b) provided a complete generalization of power GCDQ and LCMQ matrices defined on q-ordered GCD-closed sets over Euclidean domains. They established structure theorems, derived formulas for determinants, reciprocals, inverses, and analyzed p-norms for these matrices. To illustrate their results, examples were presented in the Euclidean domain of Gaussian integers, $\mathbb{Z}[i]$.

In this paper, we introduce Mersenne GCED matrices of size $n \times n$ constructed over non-exponential factor-closed sets. We provide a complete structural characterization of these matrices, compute their determinants, and derive explicit formulas for their reciprocals and inverses.

Definitions and Preliminaries

Let $S = \{x_1, x_2, \dots, x_n\}$ be an ordered set of distinct positive integers greater than 1 such that $x_1 < x_2 < \cdots < x_n$. Let $R = \{y_1, y_2, \dots, y_m\}$ denote the minimal exponential divisor-closed set containing S (i.e., the exponential closure of S), where $y_1 < y_2 < \cdots < y_m$. The $n \times n$ greatest common exponential divisor (GCED) matrix $A_e = (a_{ij})$ is defined by its entries:

$$a_{ij} = (x_i, x_j)_e.$$

Note that GCED matrices are symmetric.

Definition 1. The Möbius function, denoted as $\mu(n)$, is a number-theoretic function defined for positive integers n as follows:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n \text{ is squarefree and has } k \text{ distinct prime factors} \\ 0, & \text{if } n \text{ is not squarefree.} \end{cases}$$

Definition 2. The exponential Möbius Function, denoted as $\mu^{(e)}(n)$, is a variant of the classical Möbius function that arises in the study of exponential divisibility. It is defined as follows:

$$\mu^{(e)}(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n \text{ is exponentially squarefree and has } k \text{ distinct prime factors} \\ 0, & \text{if } n \text{ is not exponentially squarefree.} \end{cases}$$

Definition 3. The arithmetic function $g(n)$ is defined as as follows:

$$g(n) = \sum_{d \,|_e\, n} \mu^{(e)}\left(\frac{n}{d}\right).$$

Note that if n has the prime factorization $n = p_1^{c_1} p_2^{c_2} \dots p_r^{c_r}$, then $g(n)$ can be expanded as:

$$g(n) = \sum_{a_1 b_1 = c_1} \sum_{a_2 b_2 = c_2} \cdots \sum_{a_r b_r = c_r} p_1^{a_1} p_2^{a_2} \dots p_r^{a_r} \mu^{(e)}(p_1^{b_1} p_2^{b_2} \dots p_r^{b_r}).$$

The Mersenne exponential arithmetic function on S is defined as:

$$g(n) = \sum_{a_1 b_1 = c_1} \sum_{a_2 b_2 = c_2} \cdots \sum_{a_r b_r = c_r} \left(2^{p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}} - 1\right) \mu^{(e)}(p_1^{b_1} p_2^{b_2} \dots p_r^{b_r}).$$

Using Möbius inversion, we derive the Mersenne exponential formula:

$$2^n - 1 = \sum_{d \,|_e\, n} g(d).$$

Matrix Definitions

The concept of the Mersenne GCED matrices is introduced in Definition 4. Mersenne matrices combine the elegance of Mersenne numbers with the utility of structured matrices, offering interesting mathematical properties and applications in computation and cryptography.

Definition 4. The Mersenne GCED matrix M defined on S is the $n \times n$ matrix whose $(ij)^{th}$ entry is given by:

$$m_{ij} = 2^{(x_i, x_j)_e} - 1,$$

where r is any real number.

Definition 5. The incidence matrix $E_{n\times n} = e_{ij}$ of S is defined as:

$$e_{ij} = \begin{cases} 1 & \text{if } x_j \mid_e x_j, \\ 0 & \text{otherwise.} \end{cases}$$

Structure of Mersenne GCED Matrices

A complete characterization of the Mersenne GCED Matrices factorizations is given in this section.

Theorem 1. (Mersenne Structure) The Mersenne GCED matrix admits the factorization

$$M = EAE^T,$$

where:

- E is the $n \times m$ incidence matrix of R relative to S, with entries

$$e_{ij} = \begin{cases} 1 & \text{if } y_j \mid_e x_i, \\ 0 & \text{otherwise.} \end{cases}$$

- A is the $m \times m$ diagonal matrix with $a_{ii} = g(y_i)$.

Proof. By construction,

$$(EAE^T)_{ij} = \sum_{k=1}^{m} e_{ik} a_{kk} e_{jk} = \sum_{y_k \mid_e (x_i,x_j)_e} g(y_k) = 2^{(x_i,x_j)_e} - 1 = m_{ij}.$$

Theorem 2. The matrix M can also be expressed as

$$M = A_r E^T,$$

where A_r is defined by

$$a_{ij} = \begin{cases} g(y_j) & \text{if } y_j \mid_e x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Direct computation yields

$$(A_r E^T)_{ij} = \sum_{k=1}^{m} a_{ik} e_{jk} = \sum_{y_k \mid_e (x_i,x_j)_e} g(y_k) = m_{ij}.$$

Theorem 3. The matrix M factors as

$$M = AA^T,$$

Where

$$a_{ij} = \begin{cases} \sqrt{g(y_j)} & \text{if } y_j \mid_e x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Follows immediately from Theorem 1 by taking $A = E\sqrt{A}$.

Determinant of Mersenne GCED Matrices

The determinant of the Mersenne GCED Matrices is given in the next theorem.

Theorem 4. The determinant of M is given by

$$\det(M) = \sum_K \left(\left(\det\left(E(K)\right)\right)^2 \prod_{i=1}^{n} g\left(y_{k_i}\right) \right),$$

where $K = (k_1, k_2, \dots, k_n)$ ranges over all multi-indices with $1 \leq k_1 < \cdots < k_n \leq m$, and $E(K)$ is the submatrix of E formed by columns $(k_1, k_2, \dots, k_n)$.

Proof. Let $C = EA^{1/2}$. Applying the Cauchy-Binet formula to $M = CC^T$:

$$\det(M) = \sum_K \left(\det\left(C(K)\right)\right)^2 = \sum_K \left(\det\left(E(K)\right) \sqrt{\prod_{i=1}^{n} g\left(y_{k_i}\right)} \right)^2 .$$

Corollary 1. If S is exponentially factor-closed, then

$$\det(M) = \prod_{i=1}^{n} g(x_i) .$$

Reciprocal of Mersenne GCED Matrices

The reciprocals of Mersenne GCED matrices defined on S and their factorizations are studied in this section.

Definition 6. The reciprocal matrix N has entries

$$n_{ij} = \frac{1}{2^{(x_i,x_j)_e} - 1}.$$

Definition 7. The reciprocal Mersenne function h is defined inductively for $1 \leq i \leq n$ as:

$$h(x_i) = \sum_{d|_e x_i} n_{ij} \mu^{(e)} \left(\frac{x_i}{d}\right).$$

Theorem 5. The reciprocal matrix factors as

$$N = EBE^T,$$

where $B = \operatorname{diag}\left(h(y_1), \dots, h(y_m)\right)$.

Proof. By the reciprocal Mobius inversion formula:

$$(EBE^T)_{ij} = \sum_{y_k|_e (x_i,x_j)_e} h(y_k) = \frac{1}{2^{(x_i,x_j)_e} - 1}.$$

Inverse of Mersenne GCED Matrices

In this section, we investigate the Inverses of Mersenne matrices defined on a set *S*.

Definition 8. The inverse of the matrix *M*, M^{-1}, satisfies $MM^{-1} = I_n$, where I_n is the $n \times n$ identity matrix.

Theorem 6. For an exponentially factor-closed set *S*, the inverse is

$$M^{-1} = FA^{-1}F^T,$$

where:

- F has entries

$$f_{ij} = \begin{cases} \mu^{(e)}\left(\frac{x_i}{x_j}\right) & \text{if } x_j \mid_e x_j, \\ 0 & \text{otherwise.} \end{cases}$$

- A is as in Theorem 1.

Proof. Since $EF^T = I_n$ (verified via Möbius inversion), the result follows from Theorem 1.

Example

Let $S = \{4, 8, 16\}$. The Mersenne GCED matrix *M* on the set *S* is

$$M = \begin{bmatrix} 15 & 3 & 15 \\ 3 & 255 & 3 \\ 15 & 3 & 65535 \end{bmatrix}.$$

Note that the set $S = \{4, 8, 16\}$ is not exponential divisor closed. $R = \{2, 4, 8, 16\}$ is it's exponential closure set.

The matrix *E* is as follows:

$$E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

We have $g(2) = 3, g(4) = 12, g(8) = 252$, and $g(16) = 65520$.

From Theorem 1, we know that,

$$M = EAE^T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 252 & 0 \\ 0 & 0 & 0 & 65520 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 3 & 15 \\ 3 & 255 & 3 \\ 15 & 3 & 65535 \end{bmatrix}.$$

And from Theorem 2,

$$M = AE^T = \begin{bmatrix} 3 & 12 & 0 & 0 \\ 3 & 0 & 252 & 0 \\ 3 & 12 & 0 & 65520 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 3 & 15 \\ 3 & 255 & 3 \\ 15 & 3 & 65535 \end{bmatrix}.$$

Also, from Theorem 3,

$$M = AA^T = \begin{bmatrix} \sqrt{3} & \sqrt{12} & 0 & 0 \\ \sqrt{3} & 0 & \sqrt{252} & 0 \\ \sqrt{3} & \sqrt{12} & 0 & \sqrt{65520} \end{bmatrix} \begin{bmatrix} \sqrt{3} & \sqrt{3} & \sqrt{3} \\ \sqrt{12} & 0 & \sqrt{12} \\ 0 & \sqrt{252} & 0 \\ 0 & 0 & \sqrt{65520} \end{bmatrix} = \begin{bmatrix} 15 & 3 & 15 \\ 3 & 255 & 3 \\ 15 & 3 & 65535 \end{bmatrix}.$$

By Theorem 4, the determinant of *M* is given by

$$det(M) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}^2 g(2)g(4)g(8) + \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}^2 g(2)g(4)g(16) + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}^2 g(2)g(8)g(16)$$
$$+ \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}^2 g(4)g(8)g(16)$$
$$= 250024320.$$

Conclusion

In this work, the Mersenne GCED matrices defined on defined on an exponential factor-closed and a non-exponential factor-closed set were considered. A complete characterization of their structure, determinant, trace, reciprocal and inverse was given.

Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM Journal belongs to the authors.

Conflict of Interest

* The authors declare that they have no conflicts of interest

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References

Awad, Y., Mghames, R., Chehade, H., & Zeid, W. (2019). On fermat power GCD matrices defined on special sets of positive integers. *Applied Mathematical Sciences*, *13*(8), 369-379.

Awad, A., Chehade, H., & Mghames, R. (2020). Reciprocal power GCDQ matrices and power LCMQ matrices defined on factor closed sets over Euclidean domains. *Filomat*, *34*(2), 357–363.

Awad, Y., Hindi, R., & Chehade, H. (2023). Comparative study between a novel deterministic test for mersenne prımes and the well-known primality tests. *Baghdad Science Journal*, *20*(5), 50.

Awad, Y., Mghames, R., & Chehade, H. (2023). Power GCDQ and LCMQ matrices defined on GCD-closed sets over Euclidean domains. *Palestine Journal of Mathematics*, *12*(1), 30-41.

Beslin, S., & El-Kassar, N. (1989). GCD matrices and Smith's determinant for a unique factorization domain. *Bull. Number Theory Related Topics*, *13*, 17–22.

Beslin, Scott, & Ligh, S. (1989b). Greatest common divisor matrices. *Linear Algebra and Its Applications*, *118*, 69–76.

Beslin, S., & Ligh, S. (1989a). Another generalisation of Smith's determinant. *Bulletin of the Australian Mathematical Society*, *40*(3), 413–415.

Beslin, Scott, & Ligh, S. (1992). GCD-closed sets and the determinants of GCD matrices. *Fibonacci Quart*, *30*(2), 157–160.

Chehade, H. Awad, Y., & Zeid W. (2024). On lcem matrıces over unıque factorızatıon domaıns. *The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM), 28*.

Raza, Z., & Waheed, S. A. (2015a). GCED and reciprocal GCED matrices. *Hacettepe Journal of Mathematics and Statistics*, *44*(3), 633–640.

Raza, Z., & Waheed, S. A. (2015b). GCED reciprocal LCEM matrices. *Notes on Number Theory and Discrete Mathematics*, *21*(1), 79–85.

Raza, Z., & Waheed, S. A. (2012). LCEM and reciprocal GCED matrices. *Int. J. Pure and Applied Mathematics*, *80*(5), 647–655.

Zeid, W., Chehade, H., & Rasheed, F. (2022). On GCED matrices over unique factorization domains. *Filomat*, *36*(11), 3775–3784.

Smith, H. J. S. (1875). On the value of a certain arithmetical determinant. *Proceedings of the London Mathematical Society*, *1*(1), 208–213.

Subbarao, M.V. (1972). On some arithmetic convolutions, the theory of arithmetic functions, *Lecture Notes in Mathematics, 251*(1972), 247-271.

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