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Optimal Control Problem and Integral Quality Criteria with a Special Gradient Term

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Abstract: The paper deals with the optimal control problem with the full-range integral performance criterion for the nonlinear Schrödinger equation with the specific gradient summand and the complex potential when the performance criterion is the full-range integral. Concurrently, the first variation formula of the performance criterion under consideration and via which the necessary condition is established in the form of a variation inequality is found. This article investigates the differentiability of the optimal control problem for a three-dimensional nonlinear no stationary Schrödinger equation with a special gradient term and a real-valued potential, when the potential depending on both spatial and time dependence and is sought in the class of measurable bounded functions, and the coefficient in the nonlinear part of the equation is a complex number. Therefore, this work differs from previously studied problems nonlinear Schrödinger equation with a specific gradient term, complex potential and is relevant in its significance and also of scientific interest.

Keywords: Schrödinger equation, Optimal control problem, Complex potential

Introduction

Optimal control problems for various equations were previously studied in detail in Diveev et al., (2021), Emilio et al., (2022), Najafova, (2022), Salmanov (2020), Yagubov et al., (2017), etc. As is known, the Schrödinger equation with a special gradient term and initial-boundary value problems for this equation arise in quantum mechanics, nuclear physics, nonlinear optics and other areas of modern physics and technology (Iskenderov et al., 2019; Yagub et al., 2022). Especially in quantum mechanics and nonlinear optics, when studying the motion of charged particles in an inhomogeneous medium, the Schrödinger equation with a special gradient term arises, and the study of initial boundary value problems for this equation is of interest from both theoretical and practical points of view. It should be noted that the initial boundary value problems for the linear and nonlinear Schrödinger equations in various formulations were previously studied in detail in the works of others Yagub et al. (2015), Ibrahimov et al. (2022), Yakub et al. (2021), Zengin et al. (2021). However, the initial boundary value problems even for the linear Schrödinger equation with a special gradient term have been studied relatively little (Yagub et al., 2021). It should be noted that the work (Yagub et al., (2021)) study questions of the existence and uniqueness of solutions to initial boundary value problems for linear one-dimensional and two-dimensional Schrödinger equations with a special gradient term, when the coefficients are quadratically summable functions. Note that initial boundary value problems for the nonlinear Schrödinger equation with a special gradient term have been studied relatively little. In this direction, we can note the works Yagub et al. (2015), Ibrahimov et al. (2022), Yakub et al. (2021), Zengin et al. (2021) in which initial-boundary value problems for the nonlinear Schrodinger equation with a special gradient term are studied, when the coefficients of the equation depend only on a spatial variable or a time variable. Therefore, this paper is devoted to the study of the optimal control problem for a three-dimensional nonlinear Schrödinger equation with a specific gradient summand and with a complex potential, when the controls are real and imaginary parts of the complex potential and chosen from the class of measurable bounded functions depending on a time variable, and the quality criterion is the full-range integral, and it is of considerable scientific interest.

Method

Let's consider the functional minimizing problem:

$$J_\alpha(v) = \|\psi_1 - \psi_2\|^2_{L_2(\Omega)} + \alpha \|v - \omega\|^2_H \tag{1}$$

at the set:

$$V = \left\{ v = v(t) = (v_0(t), v_1(t)) :\ v_m \in W_2^1(0,T), |v_m(t)| \le b_m, \left| \frac{dv_m(t)}{dt} \right| \le d_m, m = 0,1, \overset{0}{\forall} t \in (0,T) \right\}$$

subject to the conditions that:

$$i\frac{\partial \psi_p}{\partial t} + a_0 \Delta \psi_p + i a_1(x,t) \nabla \psi_p - a(x)\psi_p + v_0(t)\psi_p + i v_1(t)\psi_p +$$

$$+ a_2 |\psi_p|^2 \psi_p = f_p(x,t), p = 1,2,\ (x,t) \in \Omega, \tag{2}$$

$$\psi_p(x,0) = \varphi_p(x),\ \text{p=1,2}, x \in D, \tag{3}$$

$$\psi_1|_S = 0, \frac{\partial \psi_2}{\partial \nu}\Big|_S = 0, \tag{4}$$

where, $i = \sqrt{-1}$; $T > 0,\ b_m > 0, d_m > 0,\ m = 0,1, a_0 > 0,\ \alpha \ge 0$ are the predetermined numbers, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the Laplace operator, $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) -$ is the nabla operator; ν is the external normal of the boundary Γ of the area D; the complex number a_2 satisfies the condition:

$$a_2 = \mathrm{Re}\, a_2 + i\, \mathrm{Im}\, a_2, \mathrm{Re}\, a_2 < 0, \mathrm{Im}\, a_2 > 0, \mathrm{Im}\, a_2 \ge 2|\mathrm{Re}\, a_2|; \tag{5}$$

$a(x)$ is a measurable bounded function that satisfies the condition:

$$\mu_0 \le a(x) \le \mu_1,\ \overset{0}{\forall} x \in D,\ \mu_0, \mu_1 = const > 0; \tag{6}$$

$a_1(x,t) = (a_{11}(x,t), a_{12}(x,t), a_{13}(x,t))$ is the specified function vector with the components that satisfy the conditions:

$$|a_{1j}(x,t)| \le \mu_2,\ \left| \frac{\partial a_{1j}(x,t)}{\partial x_k} \right| \le \mu_3, \left| \frac{\partial a_{1j}(x,t)}{\partial t} \right| \le \mu_4, j,k = \overline{1,3},$$
$$\overset{0}{\forall}(x,t) \in \Omega, a_{1j}|_S = 0, j = \overline{1,n},\ \mu_2, \mu_3, \mu_4 = const > 0; \tag{7}$$

$\varphi_p(x), f_p(x,t), p = 1,2$ are the complex-valued functions that satisfy the conditions:

$$\varphi_1 \in \overset{0}{W}{}_2^2(D), \varphi_2 \in W_2^2(D), \frac{\partial \varphi_2}{\partial \nu}\Big|_\Gamma = 0; \tag{8}$$

$$f_p \in W_2^{0,1}(\Omega), p = 1,2; \tag{9}$$

$\omega \in H$ is a specified element, where $H \equiv W_2^1(0,T) \times W_2^1(0,T)$ and the symbol $\overset{0}{\forall}$ means "at almost all". The problem of determining functions $\psi_p = \psi_p(x,t) \equiv \psi_p(x,t;v), p = 1,2$ from the conditions (2)-(4) upon every $v \in V$ will be referred to as a reduced problem.

Definition 1. Under the solution of the reduced problems (2)-(4) for each $v \in V$, we mean a functions $\psi_1 \in B_1 \equiv C^0\left(\left[0,T\right],\overset{0}{W}{}_2^2(D)\right) \cap C^1\left(\left[0,T\right],L_2(D)\right)$, $\psi_2 \in B_2 \equiv C^0\left(\left[0,T\right],W_2^2(D)\right) \cap C^1\left(\left[0,T\right],L_2(D)\right)$, for almost all $x \in D$ and $t \in [0,T]$ satisfying equations (2), and the initial conditions (3) for almost all $x \in D$ and the boundary conditions (4) for almost $(\xi,t) \in S$.

It should be noted that the reduced problem type (2)-(4), that is, the first and second initial-boundary value problems (2)-(4) were previously studied in the paper Iskenderov et al. (2018). Based on the results of this paper the following statement can be formulated:

Theorem 1. Let's assume that the complex number a_2 satisfies the condition (5), and the functions $a(x)$, $a_1(x,t)$, $\varphi_p(x)$, $f_p(x,t), p=1,2$ satisfy the conditions (6)-(9). Then, the reduced problems (2)-(4) upon every $v \in V$ have the single solution $\psi_1 \in B_1, \psi_2 \in B_2$, and the following statements are valid for such solution:

$$\left\|\psi_1(\cdot,t)\right\|^2_{\overset{0}{W}{}_2^2(D)} + \left\|\frac{\partial \psi_1(\cdot,t)}{\partial t}\right\|^2_{L_2(D)} \le c_1\left(\left\|\varphi_1\right\|^2_{\overset{0}{W}{}_2^2(D)} + \left\|f_1\right\|^2_{W_2^{0,1}(\Omega)} + \left\|\varphi_1\right\|^6_{\overset{0}{W}{}_2^1(D)}\right), \forall t \in [0,T], \quad (10)$$

$$\left\|\psi_2(\cdot,t)\right\|^2_{\overset{0}{W}{}_2^2(D)} + \left\|\frac{\partial \psi_2(\cdot,t)}{\partial t}\right\|^2_{L_2(D)} \le c_2\left(\left\|\varphi_2\right\|^2_{W_2^2(D)} + \left\|f_2\right\|^2_{W_2^{0,1}(\Omega)} + \left\|\varphi_2\right\|^6_{W_2^1(D)}\right), \forall t \in [0,T], \quad (11)$$

where, $c_p > 0, p=1,2$ are the constants not depending on t.

According to this theorem, the functional described in (1) makes sense for the problem under consideration. In work Salmanov (2020) the existence and uniqueness theorems for the solution of the optimal control problem we are considering for the nonlinear Schrödinger equation with a special gradient term and with a complex potential are proved, when the quality criterion is the integral over the entire domain.

The Functional Differentiability and the Necessary Condition for Solution of the Optimal Control Problems

Let us consider the necessary condition for solving problems (1)-(4) in the form of the variational inequality. Let's suppose that $\Phi_p = \Phi_p(x,t), p=1,2$ are the solution for the following conjugate problem:

$$i\frac{\partial \Phi_p}{\partial t} + a_0 \Delta \Phi_p + i\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(a_{1j}(x,t)\Phi_p\right) - a(x)\Phi_p + v_0(t)\Phi_p - iv_1(t)\Phi_p +$$

$$+2\bar{a}_2\left|\psi_p\right|^2\Phi_p + a_2\left(\psi_p\right)^2\bar{\Phi}_p = (-1)^p\, 2\left(\psi_1(x,t) - \psi_2(x,t)\right), p=1,2, (x,t)\in\Omega, \quad (12)$$

$$\Phi_p(x,T) = 0, p=1,2, x \in D, \quad (13)$$

$$\Phi_1\big|_S = 0, \ \left.\frac{\partial \Phi_2}{\partial \nu}\right|_S = 0, \quad (14)$$

where $\psi_p = \psi_p(x,t) \equiv \psi_p(x,t;v), p=1,2$ is a solution to (2)-(4) provided that $v \in V$.

The solution of the conjugate problems (12)-(14) means the function $\Phi_p(x,t), p=1,2$ from the space $B_p, p=1,2$ meeting the equations (12) for $\forall t \in [0,T]$, the boundary conditions (13) and (14) for $\overset{\circ}{\forall} x \in D$ and $\overset{\circ}{\forall} (\xi,t) \in S$, respectively.

Theorem 2. Let's suppose that the number a_2 and functions $a(x), a_1(x,t)$, $\varphi_p(x)$, $f_p(x,t), p=1,2$ satisfy the conditions (5)-(9). Also, let's assume that $y \in \overset{0}{W}{}_2^2(D)$ is a specified function. Then, the conjugate problem (12)-(14) has the single solution from the space $\Phi_p \in B_p, p=1,2$ and the following estimates are valid for this solution:

$$\left\|\Phi_1(.,t)\right\|^2_{\overset{0}{W}{}_2^2(D)} + \left\|\frac{\partial \Phi_1(.,t)}{\partial t}\right\|^2_{L_2(D)} \le c_5 \left\|\psi_1 - \psi_2\right\|^2_{W_2^{0,1}(\Omega)}, \tag{15}$$

$$\left\|\Phi_2(.,t)\right\|^2_{W_2^2(D)} + \left\|\frac{\partial \Phi_2(.,t)}{\partial t}\right\|^2_{L_2(D)} \le c_6 \left\|\psi_1 - \psi_2\right\|^2_{W_2^{0,1}(\Omega)} \tag{16}$$

for $\forall t \in [0,T]$, where $c_5 > 0, c_6 > 0$ are the constant not depending on t.

This theorem is proved by the Galerkin method. To establish the necessary condition for the solution by specifying (1)-(4), we must show the differentiability of the functional $J_\alpha(v)$ onto the set V.

Theorem 3. Let's assume that the hypotheses of theorem 2 are satisfied and $\omega \in H$ is a specified element. Then, the functional $J_\alpha(v)$ for any function $w = w(t)$ from the space B has the first variation at the set V as follows:

$$\begin{aligned}
\delta J_\alpha(v,w) = &\int_\Omega \operatorname{Re}\left(\psi_1(x,t)\bar{\Phi}_1(x,t) + \psi_2(x,t)\bar{\Phi}_2(x,t)\right) w_0(t)\,dxdt - \\
&-\int_\Omega \operatorname{Im}\left(\psi_1(x,t)\bar{\Phi}_1(x,t) + \psi_2(x,t)\bar{\Phi}_2(x,t)\right) w_1(t)\,dxdt + \\
&+2\alpha\int_0^T \left(v_0(t) - \omega_0(t)\right) w_0(t)\,dt + \int_0^T \left(\frac{dv_0(t)}{dt} - \frac{d\omega_0(t)}{dt}\right)\frac{dw_0(t)}{dt}dt + \\
&+2\alpha\int_0^T \left(v_1(t) - \omega_1(t)\right) w_1(t)\,dt + \int_0^T \left(\frac{dv_1(t)}{dt} - \frac{d\omega_1(t)}{dt}\right)\frac{dw_1(t)}{dt}dt,
\end{aligned} \tag{17}$$

where $\psi_p(x,t) \equiv \psi_p(x,t;v), \Phi_p(x,t) \equiv \Phi_p(x,t;v), p=1,2$ are the solutions of the reduced problem (2)-(4), respectively, and the conjugate problem (12)-(14) provided that $v \in V$.

Proof. Let's consider the increment of the functional $J_\alpha(v)$ at any element $v \in V$. Based on the formulas (1) and (16) the following is obtained:

$$\begin{aligned}
\delta J_\alpha(v) = J_\alpha(v+\delta v) - J_\alpha(v) = &\, 2\int_\Omega \operatorname{Re}\left[\left(\psi_1(x,t) - \psi_2(x,t)\right)\left(\delta\bar{\psi}_1(x,t) - \delta\bar{\psi}_2(x,t)\right)\right]dx + \\
&+2\alpha\int_0^T \left(v_0(t) - \omega_0(t)\right) v_0(t)\,dt + 2\alpha\int_0^T \left(\frac{dv_0(t)}{dt} - \frac{d\omega_0(t)}{dt}\right)\frac{dv_0(t)}{dt}dt + \\
&+2\alpha\int_0^T \left(v_1(t) - \omega_1(t)\right) v_1(t)\,dt + 2\alpha\int_0^T \left(\frac{dv_1(t)}{dt} - \frac{d\omega_1(t)}{dt}\right)\frac{dv_1(t)}{dt}dt + \\
&+\left\|\delta\psi_1\right\|^2_{L_2(\Omega)} + \left\|\delta\psi_2\right\|^2_{L_2(\Omega)} - 2\int_\Omega \operatorname{Re}\left(\delta\psi_1(x,t)\,\delta\bar{\psi}_2(x,t)\right)dxdt + \alpha\left\|\delta v\right\|^2_H, \forall v \in V,
\end{aligned} \tag{18}$$

where $\delta\psi_p = \delta\psi_p(x,t), p=1,2$ is the solution of the system of the following initial boundary value problems:

$$i\frac{\partial\delta\psi_p}{\partial t} + a_0\Delta\psi_p + ia_1(x,t)\nabla\delta\psi_p - a(x)\delta\psi_p + (v_0(t)+\delta v_0(t))\delta\psi_p + i(v_1(t)+\delta v_1(t))\delta\psi_p =$$

$$= -\delta v_0(t)\psi_p - i\delta v_1(t)\psi_p - a_2\left(|\psi_{p\delta}|^2\psi_{p\delta} - |\psi_p|^2\psi_p\right), p=1,2\ (x,t)\in\Omega$$

$$\delta\psi_p(x,0)=0, p=1,2\ x\in D\ \delta\psi_1|_S = 0, \frac{\partial\delta\psi_2}{\partial\nu}\bigg|_S = 0$$

At first, the first summand from the right part of the formula is to be rearranged.

It is obvious that $\delta\psi_p \in B_p, p=1,2$ satisfy the following integral equalities:

$$\int_\Omega\left(i\frac{\partial\delta\psi_p}{\partial t} + a_0\Delta\delta\psi_p + ia_1(x,t)\nabla\delta\psi_p - a(x)\delta\psi_p + (v_0(t)+\delta v_0(t))\delta\psi_p\right)\overline{\eta}_p(x,t)dxdt +$$

$$+i\int_\Omega(v_1(t)+\delta v_1(t))\delta\psi_p\overline{\eta}_p(x,t)dxdt = -\int_\Omega\delta v_0(t)\psi_p\overline{\eta}_p(x,t)dxdt -$$

$$-i\int_\Omega\delta v_1(t)\psi_p\overline{\eta}_p(x,t)dxdt - \int_\Omega a_2\left[\left(|\psi_{p\delta}|^2 + |\psi_p|^2\right)\delta\psi_p + \psi_{p\delta}\psi_p\delta\overline{\psi}_p\right]\overline{\eta}_p(x,t)dxdt, p=1,2 \quad (19)$$

for any functions $\eta_p \in L_2(\Omega), p=1,2$. Also, the solution of the conjugate problem $\Phi_p(x,t), p=1,2$ from $B_p, p=1,2$ satisfies the following integral equalities:

$$\int_\Omega\left(i\frac{\partial\Phi_p}{\partial t} + a_0\Delta\Phi_p + i\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(a_{1j}(x,t)\Phi_p\right) - a(x)\Phi_p + v_0(t)\Phi_p - iv_1(t)\Phi_p +\right.$$

$$\left.+2\overline{a}_2|\psi_p|^2\Phi_p + a_2(\psi_p)^2\overline{\Phi}_p\right)\overline{\eta}_{1p}(x,t)dxdt = (-1)^p 2\int_\Omega(\psi_1(x,t)-\psi_2(x,t))\overline{\eta}_{1p}(x,t)dxdt \quad 20)$$

for any functions $\eta_{1p} \in L_2(\Omega), p=1,2$. In such integral equalities, the testing functions $\eta_{1p}(x,t), p=1,2$ are replaced with the functions $\delta\psi_p(x,t), p=1,2$ from $B_p, p=1,2$, respectively. Then, after applying integration by parts of the left side of the equality we obtained using conditions of the form (14) and (12), we obtain an equality whose complex conjugation has the form:

$$\int_\Omega\left(i\frac{\partial\delta\psi_p}{\partial t} + a_0\Delta\delta\psi_p + ia_1(x,t)\nabla\delta\psi_p - a(x)\delta\psi_p + v_0(t)\delta\psi_p + iv_1(t)\delta\psi_p\right)\overline{\Phi}_p dxdt +$$

$$+\int_\Omega 2a_2|\psi_p|^2\delta\psi_p\overline{\Phi}_p dxdt + \int_\Omega\overline{a}_2(\overline{\psi}_p)^2\delta\psi_p\Phi_p dxdt =$$

$$= (-1)^p 2\int_D(\overline{\psi}_1(x,t)-\overline{\psi}_2(x,t))\delta\psi_p(x,t)dxdt, p=1,2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

In the integral equality (19), the testing functions $\eta_p(x,t), p=1,2$ are replaced with $\Phi_p(x,t), p=1,2$ from $B_p, p=1,2$, respectively. Then, the following is found:

$$\int_{\Omega}\left(i\frac{\partial\delta\psi_p}{\partial t}+a_0\Delta\delta\psi_p+ia_1(x,t)\nabla\delta\psi_p-a(x)\delta\psi_p+\left(v_0(t)+\delta v_0(t)\right)\delta\psi_p\right)\bar{\Phi}_p(x,t)\,dxdt+$$

$$+i\int_{\Omega}\left(v_1(t)+\delta v_1(t)\right)\delta\psi_p\bar{\Phi}_p(x,t)\,dxdt=-\int_{\Omega}\delta v_0(t)\psi_p\bar{\Phi}_p(x,t)\,dxdt-$$

$$-i\int_{\Omega}\delta v_1(t)\psi_p\bar{\Phi}_p(x,t)\,dxdt-\int_{\Omega}a_2\left[\left(\left|\psi_{p\delta}\right|^2+\left|\psi_p\right|^2\right)\delta\psi_p+\psi_{p\delta}\psi_p\delta\bar{\psi}_p\right]\bar{\Phi}_p(x,t)\,dxdt,\ p=1,2$$

Through deducting the equity (21), the validity if the equity is easily found:

$$2\int_{\Omega}\mathrm{Re}\left[\left(\psi_1(x,t)-\psi_2(x,t)\right)\left(\delta\bar{\psi}_1(x,t)-\delta\bar{\psi}_2(x,t)\right)\right]dxdt=\int_{\Omega}\delta v_0(t)\,\mathrm{Re}\left(\sum_{p=1}^{2}\psi_p\bar{\Phi}_p\right)dxdt-$$

$$-\int_{\Omega}\delta v_1(t)\,\mathrm{Im}\left(\sum_{p=1}^{2}\psi_p\bar{\Phi}_p\right)dxdt+\int_{\Omega}\delta v_0(t)\,\mathrm{Re}\left(\sum_{p=1}^{2}\delta\psi_p\bar{\Phi}_p\right)dxdt-$$

$$-\int_{\Omega}\delta v_1(t)\,\mathrm{Im}\left(\sum_{p=1}^{2}\delta\psi_p\bar{\Phi}_p\right)dxdt+\int_{\Omega}\mathrm{Re}\left(a_2\sum_{p=1}^{2}\psi_{p\delta}\bar{\Phi}_p\left|\delta\psi_p\right|^2\right)dxdt+$$

$$+\int_{\Omega}\mathrm{Re}\left(a_2\sum_{p=1}^{2}\psi_p\bar{\Phi}_p\left|\delta\psi_p\right|^2\right)dxdt+\int_{\Omega}\mathrm{Re}\left(a_2\sum_{p=1}^{2}\bar{\psi}_p\bar{\Phi}_p\left(\delta\psi_p\right)^2\right)dxdt\,. \tag{22}$$

Having considered the equity in the right part (18), the following is obtained:

$$\delta J_\alpha(v)=\int_{\Omega}\delta v_0(t)\,\mathrm{Re}\left(\sum_{p=1}^{2}\psi_p\bar{\Phi}_p\right)dxdt-\int_{\Omega}\delta v_1(t)\,\mathrm{Im}\left(\sum_{p=1}^{2}\psi_p\bar{\Phi}_p\right)dxdt+$$

$$+2\alpha\int_0^T\left(v_0(t)-\omega_0(t)\right)\delta v_0(t)\,dt+2\alpha\int_0^T\left(\frac{dv_0(t)}{dt}-\frac{d\omega_0(t)}{dt}\right)\frac{d\delta v_0(t)}{dt}dt+$$

$$+2\alpha\int_0^T\left(v_1(t)-\omega_1(t)\right)\delta v_1(t)\,dt+2\alpha\int_0^T\left(\frac{dv_1(t)}{dt}-\frac{d\omega_1(t)}{dt}\right)\frac{d\delta v_1(t)}{dt}dt+R(\delta v), \tag{23}$$

where $R(\delta v)$ is specified through the following formula:

$$R(\delta v)=\int_{\Omega}\delta v_0(t)\,\mathrm{Re}\left(\sum_{p=1}^{2}\delta\psi_p\bar{\Phi}_p\right)dxdt-\int_{\Omega}\delta v_1(t)\,\mathrm{Im}\left(\sum_{p=1}^{2}\delta\psi_p\bar{\Phi}_p\right)dxdt+$$

$$+\int_{\Omega}\mathrm{Re}\left(a_2\sum_{p=1}^{2}\psi_{p\delta}\bar{\Phi}_p\left|\delta\psi_p\right|^2\right)dxdt+\int_{\Omega}\mathrm{Re}\left(a_2\sum_{p=1}^{2}\psi_p\bar{\Phi}_p\left|\delta\psi_p\right|^2\right)dxdt+$$

$$+\int_{\Omega}\mathrm{Re}\left(a_2\sum_{p=1}^{2}\bar{\psi}_p\bar{\Phi}_p\left(\delta\psi_p\right)^2\right)dxdt+\alpha\left\|\delta v\right\|_H^2+$$

$$+\sum_{p=1}^{2}\left\|\delta\psi_p\right\|_{L_2(\Omega)}^2-2\int_{\Omega}\mathrm{Re}\left(\delta\psi_1\delta\bar{\psi}_2\right)dxdt,\ \forall v\in V \tag{24}$$

Upon the estimation of the rest summand $R(\delta v)$, then, through the Cauchy-Bunyakovsky inequality, the following is valid:

$$|R(\delta v)| \le \left(\|\delta v_0\|_{L_\infty(0,T)} + \|\delta v_1\|_{L_\infty(0,T)}\right)\sum_{p=1}^{2}\|\Phi_p\|_{L_2(\Omega)}\|\delta\psi_p\|_{L_2(\Omega)} +$$

$$+4|a_2|\sum_{p=1}^{2}\left(\|\psi_{p\delta}\|_{L_\infty(\Omega)} + \|\psi_p\|_{L_\infty(\Omega)}\right)\|\Phi_p\|_{L_\infty(\Omega)}\|\delta\psi_p\|^2_{L_2(\Omega)} +$$

$$+\alpha\|\delta v\|^2_H + 2\sum_{p=1}^{2}\|\delta\psi_p\|^2_{L_2(\Omega)}. \qquad (25)$$

In view of inclusion of the space $W_2^2(D)$ into $L_\infty(D)$ [22] provided that $n = 3$ the following are found:

$$\|\psi_{p\delta}(.,t)\|_{L_\infty(D)} \le c_7\|\psi_{p\delta}(.,t)\|_{W_2^2(D)}, p = 1,2, \qquad (26)$$

$$\|\psi_p(.,t)\|_{L_\infty(D)} \le c_8\|\psi_p(.,t)\|_{W_2^2(D)}, p = 1,2, \qquad (27)$$

$$\|\Phi_p(.,t)\|_{L_\infty(D)} \le c_9\|\Phi_p(.,t)\|_{W_2^2(D)}, p = 1,2 \qquad (28)$$

for any $t \in [0,T]$. Using the estimates (10), (11) for the functions $\psi_{p\delta}(x,t)$, $\psi_p(x,t), p = 1,2$ and the estimates (15), (16) for the functions $\Phi_p(x,t), p = 1,2$ the following inequality is valid:

$$\|\psi_{p\delta}\|_{L_\infty(\Omega)} \le c_{10}, \ \|\psi_p\|_{L_\infty(\Omega)} \le c_{11}, \ \|\Phi_p\|_{L_\infty(\Omega)} \le c_{12}, p = 1,2. \qquad (29)$$

Subject to such inequalities and the estimates (15), (15), (16) from (25), the following is found:

$$|R(\delta v)| \le c_{13}\|\delta v\|^2_B. \qquad (30)$$

It means that

$$R(\delta v) = o(\|\delta v\|_B). \qquad (31)$$

Then, in view of such correlation, the increment $J_\alpha(v)$ may be recorded as follows:

$$\delta J_\alpha(v) = J_\alpha(v+\delta v) - J_\alpha(v) = \int_\Omega \delta v_0(t)\,\mathrm{Re}\left(\psi_1(x,t)\bar{\Phi}_1(x,t) + \psi_2(x,t)\bar{\Phi}_2(x,t)\right)dxdt -$$

$$-\int_\Omega \delta v_1(t)\,\mathrm{Im}\left(\psi_1(x,t)\bar{\Phi}_1(x,t) + \psi_2(x,t)\bar{\Phi}_2(x,t)\right)dxdt +$$

$$+2\alpha\int_0^T\left(v_0(t) - \omega_0(t)\right)v_0(t)dt + 2\alpha\int_0^T\left(\frac{dv_0(t)}{dt} - \frac{d\omega_0(t)}{dt}\right)\frac{dv_0(t)}{dt}dt +$$

$$+2\alpha\int_0^T\left(v_1(t) - \omega_1(t)\right)v_1(t)dt + 2\alpha\int_0^T\left(\frac{dv_1(t)}{dt} - \frac{d\omega_1(t)}{dt}\right)\frac{dv_1(t)}{dt}dt + o(\|\delta v\|_B), \forall v \in V. \qquad (32)$$

In this equity, $\delta v \in B$ is replaced with $\theta w \in B$, where $0 < \theta < 1$ and $w = w(t)$ are any function from the space B. Then, according to (32), the following is found:

$$\delta J_\alpha(v) = J_\alpha(v+\theta w) - J_\alpha(v) = \theta\int_\Omega \mathrm{Re}\left(\psi_1(x,t)\bar{\Phi}_1(x,t) + \psi_2(x,t)\bar{\Phi}_2(x,t)\right)w_0(t)\,dxdt -$$

$$-\theta\int_\Omega \mathrm{Im}\left(\psi_1(x,t)\bar{\Phi}_1(x,t) + \psi_2(x,t)\bar{\Phi}_2(x,t)\right)w_1(t)\,dxdt +$$

$$+2\alpha\theta\int_0^T \left(v_0(t)-\omega_0(t)\right)w_0(t)\,dt + 2\alpha\theta\int_0^T \left(\frac{dv_0(t)}{dt} - \frac{d\omega_0(t)}{dt}\right)\frac{dw_0(t)}{dt}dt +$$

$$+2\alpha\theta\int_0^T \left(v_1(t)-\omega_1(t)\right)w_1(t)\,dt + 2\alpha\theta\int_0^T \left(\frac{dv_1(t)}{dt} - \frac{d\omega_1(t)}{dt}\right)\frac{dw_1(t)}{dt}dt + o(\theta), \forall v \in V. \quad (33)$$

Deducting the first variation of the functional through this formula, the following formula is validated:

$$\delta J_\alpha(v,w) = \lim_{\theta\to 0}\frac{J_\alpha(v+\theta w) - J_\alpha(v)}{\theta} = \int_\Omega \mathrm{Re}\left(\psi_1(x,t)\bar{\Phi}_1(x,t) + \psi_2(x,t)\bar{\Phi}_2(x,t)\right)w_0(t)\,dxdt -$$

$$-\int_\Omega \mathrm{Im}\left(\psi_1(x,t)\bar{\Phi}_1(x,t) + \psi_2(x,t)\bar{\Phi}_2(x,t)\right)w_1(t)\,dxdt +$$

$$+2\alpha\int_0^T \left(v_0(t)-\omega_0(t)\right)w_0(t)\,dt + 2\alpha\int_0^T \left(\frac{dv_0(t)}{dt} - \frac{d\omega_0(t)}{dt}\right)\frac{dw_0(t)}{dt}dt +$$

$$+2\alpha\int_0^T \left(v_1(t)-\omega_1(t)\right)w_1(t)\,dt + 2\alpha\int_0^T \left(\frac{dv_1(t)}{dt} - \frac{d\omega_1(t)}{dt}\right)\frac{dw_1(t)}{dt}dt, \forall v \in V \quad (34)$$

for any function $w \in B$. Hence, the conclusion of the theorem is found. So, theorem 3 is proved. Using the theorem above, we now prove the necessary condition in the form of a variational inequality:

Theorem 4. Let us assume that all the conditions of theorem 3 are met. We assume that $v^* \in V$ -is any solution of the optimal control problem (1)-(3). Then, for any $v \in V$ the following inequality is valid:

$$\int_0^T \left[\int_D \mathrm{Re}\left(\psi_1^*(x,t)\bar{\Phi}_1^*(x,t) + \psi_2^*(x,t)\bar{\Phi}_2^*(x,t)\right)dx + 2\alpha\left(v_0^*(t)-\omega_0(t)\right)\right]\left(v_0(t)-v_0^*(t)\right)dt +$$

$$\int_0^T \left[\int_D -\mathrm{Im}\left(\psi_1^*(x,t)\bar{\Phi}_1^*(x,t) + \psi_2^*(x,t)\bar{\Phi}_2^*(x,t)\right)dx + 2\alpha\left(v_1^*(t)-\omega_1(t)\right)\right]\left(v_1(t)-v_1^*(t)\right)dt +$$

$$+2\alpha\int_0^T \left(\frac{dv_0^*(t)}{dt} - \frac{d\omega_0(t)}{dt}\right)\left(\frac{dv_0(t)}{dt} - \frac{dv_0^*(t)}{dt}\right)dt +$$

$$+2\alpha\int_0^T \left(\frac{dv_1^*(t)}{dt} - \frac{d\omega_1(t)}{dt}\right)\left(\frac{dv_1(t)}{dt} - \frac{dv_1^*(t)}{dt}\right)dt \geq 0, \quad (35)$$

where $\psi_p^*(x,t) \equiv \psi_p(x,t;v^*)$, $\Phi_p^*(x,t) = \Phi_p(x,t;v^*), p = 1,2$ - is the solution of reduced initial boundary (2)-(4) and conjugate problems (12)-(14) provided that $v^* \in V$.

Proof. Let's suppose that $v \in V$ is a certain element and $v^* \in V$ is an arbitrary solution for the optimal control problem (1)-(4). According to the structure of the set V, it is a convex set. So, for $v^* \in V$ and any $v \in V$ the following is found:

$$v^* + \theta\left(v - v^*\right) \in V, \forall \theta \in (0,1).$$

Therefore, to consider $v^* \in V$ as a minimal point of the functional $J_\alpha(v)$ at the set V, the satisfaction of the inequality for any $v \in V$ is necessary:

$$\left.\frac{d}{d\theta} J_\alpha\left(v^* + \theta\left(v - v^*\right)\right)\right|_{\theta=0} = \delta J_\alpha\left(v^*, v - v^*\right) \geq 0 .$$

Hence, according to formula (17) provided that $w(t) = v(t) - v^*(t)$, the conclusion of the theorem is found. So, theorem 4 is proved.

Previously, initial boundary value problems for the nonlinear Schrödinger equation with a special gradient term were studied, when the coefficients of the equation depend only on a spatial variable or a time variable. In this paper, we study the problems of optimal control of a three-dimensional nonlinear Schrödinger equation with a specific gradient term and a complex potential, when the controls are the real and imaginary parts of the complex potential and are selected from the class of measurable bounded functions that depend on a time variable and the quality criterion is the full-range integral, which has not been previously studied.

Conclusion

The resolvability theorems as proved above, and the reported expression for the first variation of the quality criterion, as well as the displayed essential extremum condition enable to apply the numerical methods for solution of incorrect and inverse problems, including the optimal control problems with crude data arisen upon studying the movement process of charged particles in the constant uniform magnetic field where the complex potential is unknown and is to be determined.

Recommendations

The results of this paper can be used by researchers for the optimal control problem of a three-dimensional nonlinear Schrödinger equation with a specific gradient term and complex potential.

Scientific Ethics Declaration

The author declares that the scientific ethical and legal responsibility of this article published in EPSTEM Journal belongs to the author.

Conflict of Interest

The author declare that they have no conflicts of interest.

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