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Notes to the Question of Presenting the Theme of Special Solutions of Ordinary Differential Equations in a University Course

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Abstract: As Sir Isaac Newton has said, laws of the Nature have been written in the language of Differential Equations. In particular, the classical theory of normal systems of Ordinary Differential Equations, supported by Cauchy theorems of existence and uniqueness of solutions, describes determined processes taking place in the Nature, technics and even in the society, i.e. such processes, for which a condition of a described system in an arbitrary fixed moment depends on its condition in any other moment. Solutions, describing such processes, are called the ordinary. But when the conditions of the Cauchy theorem are not satisfied, a situation totally changes. A point, in any neighborhood of which such conditions are not satisfied, may become for a system under consideration a point of non-uniqueness, a point of bifurcation. A solution of a system, each point of which appears to be a point of non-uniqueness, is called a special solution. A task of a full integration of a system demands finding of all its solutions, special solutions as well as ordinary ones. But this item shows us some gap in a special literature. This paper presents materials with the aim to fill this gap.

Keywords: Differential equations, Ordinary solution, Special solution, Bifurcations

Introduction

Let us consider a differential equation of the first order which is not resolved with respect to the derivative

$F\,(x, y, y') = 0,$ (1)

where F and $F'_{y'} \in C(D), D\ (\subset \mathbb{R}^3)$ is a domain, and $F'_{y'}\,(x, y, y') \not\equiv 0$ in any domain $U \subset D$, (otherwise in such a domain U function F will not depend on y', and consequently equation (1) won`t be a differential equation in U).

The *solution* of Eq. (1) is called any function belonging to the C^1 class (that means it must have a continuous derivative)

$$\varphi : I = \langle \alpha, \beta \rangle \to \mathbb{R},$$

(where I be an open, semi open or non-degenerate interval of the x axis), which is, being substituted into the Eq. (1) instead of y, transforms Eq. (1) into an identity with respect to x: $F(x, \varphi(x), \varphi'(x)) \equiv 0, x \in I.$

Any solution graph is called the *integral curve* of this equation.

A *region of definition* of Eq. (1) is called a set G belonging to an *(x, y)* plane, such as for any $p_0 = (x_0, y_0) \in G$ the equation

$F\,(x_0, y_0, y') = 0$ (2)

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has at least one solution $y' = y'_0, q_0 = (x_0, y_0, y'_0) \in D$.

Geometrically this means, that Eq. (1) sets in any point $p_0 \in G$ at least one tangential direction y'_0 for its integral curves. For the Eq. (1) the solutions of the Eq. (2) are called *allowable values* of y' in a p_0 point.

A *Cauchy problem* for the Eq. (1) is being formulated in the following way: we set a point p_0=(x_0, y_0) $\in G$ and an allowed for it value $y' = y'_0$; it`s necessary to find a solution φ of the Eq. (1), satisfying conditions

$$\varphi(x_0) = y_0, \varphi'(x_0) = y'_0. \qquad (3)$$

A point $q_0 = (x_0, y_0, y'_0)$ is called the *initial point* of the solution φ and of the integral curve corresponding to this solution $y = \varphi(x)$.

Conditions (3) are called the *initial conditions* (or the *initial data*) for the solution φ. The solution of the Cauchy problem is called *unique*, if for every two of its solutions $\varphi_1(x)$, $x \in I_1$, $\varphi_2(x)$, $x \in I_2$, $\exists \partial > 0$: $\varphi_2(x) \equiv \varphi_1(x)$ for $x \in I_1 \cap I_2 \cap (x_0 - \partial, x_0 + \partial)$. In an opposite case such a solution is called *nonunique*.

A solution of Eq. ($y = \varphi(x), x \in I$, is called *ordinary* (*special*), if for each its point $q_0 = (x_0, y_0, y'_0) = (x_0, \varphi(x_0), \varphi'(x_0))$ the solution of a Cauchy problem Eq. (1), Eq. (3) is *unique* (*non-unique*).

An integral curve of the Eq. (1) is called *ordinary* (*special*), if it represents a graph of an *ordinary* (*special*) solution.

A point p_0= (x_0, y_0) $\in G$ is called *a point of uniqueness for Eq. (1)*, if for each allowable for it value $y' = y'_0$ a solution of the Cauchy problem Eq.(1), Eq. (3) is unique or doesn`t exist. In the opposite case it`s called for eq. (1) a *point of non-uniqueness*.

The relation

$$\Phi(x, y, C) = 0 \qquad (4)$$

where *C* means an arbitrary constant, is called a *general integral* for Eq. (1) on a set G' ($\subset G$), if for every point p_0=(x_0, y_0) $\in G'$ and any allowed for it value $y' = y'_0$ it implicitly defines a unique solution of the Caushy problem.

Using a general integral (4) of the Eq. (1), we sometimes are able to find the special integral curves which lie in G' , as envelopes of a family of curves defined with this integral.

The Sufficient Attribute of the Envelope

As it was shown by V. Zalgaller (1975) and A. Andreev, I. Andreeva (2002), the following theorem takes place.

Theorem 1.

Let

$$\Phi(x, y, C) = 0,\ \Phi \in C^2(D),\ (D \in \mathbb{R}^3 - \text{a domain}), \qquad (5)$$

be a family of curves, depending on a parameter $C \in I = (a, b)$ and covering the set *G* at the *(x, y)* plane, *G x I* $\subset$ *D*. Let $\Phi'_y(x, y, C) \neq 0$ in *G x I.*

If the system of equations

$$\Phi(x, y, C) = 0, \quad \Phi'_C(x, y, C) = 0, \quad (x, y, C) \in GxI, \tag{6}$$

has a solution of $C^1 -$ class

$$y = \psi(x), \qquad C = C(x), \qquad x \in J, \tag{7}$$

and $C'(x) \not\equiv 0$ on every interval $J' \subset J$, then a curve $y = \psi(x),\ x \in J$, is an envelope of a family of curves (5), which is tangential in every its point $\big(x_0, \psi(x_0)\big)$ to a curve of this family $\Phi(x, y, C(x_0)) = 0$.

Corollary 1.

Let $\Phi(x, y, C) = 0$, where $C \in I = (a, b)$ is an arbitrary constant, be a general integral of the Eq. (1) on a *G* set. If for a family of curves (4) the conditions of the Theorem 1 are satisfied, and consequently this family of curves has an envelope $y = \psi(x),\ x \in J,$ then such an envelope appears to be a special integral curve for the Eq. (1).

A Proof.

An envelope of a family of integral curves of the Eq. (1) always is an integral curve for the Eq. (1) itself. The same time every point of it is a point of non-uniqueness, consequently it always appears to be a special integral curve for the Eq. (1).

Conclusions

The next stages of a consideration of the theme contain the thorough study of the algebraic and common non-algebraic cases including the parametrical integration of the equations. (A. Andreev, I. Andreeva, 2002). This study leads to formulating and proof of attributes of existing of the special solutions (I. Andreeva, 2003).

Recommendations

The article presents a review of data and results, which are insufficiently represented in general educational literature.

Methods of investigation may be useful for applied studies of ordinary differential equations having special solutions.

The paper may be interesting for researchers (A. Andreev, I. Andreeva, 2017) as well as for students and post-graduate students.

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