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Stabilized Finite Element Solution of Control Problem of Convection Diffusion Equation

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Abstract: In this work, we consider the stabilized numerical solutions of optimal control problems of convection diffusion equation of this type of equations have been commonly studied in the literature. We use finite element method (FEM). Because of the viscosity term in the problem, the FEM solution blows up if Reynold number is large. In this case, the solution is unstabilized so that a stabilization technique is needed. As for stabilization technique, we apply both variational multiscale (VMS) and grad-div stabilization technique. The variational multiscale method is reviewed as a framework for developing computational methods for large-eddy simulation of turbulent flow. Some of the most used numerical stabilization techniques for flow problems are streamline upwind Galerkin (SUPG) and pressure stabilization methods, large eddy simulation (LES) methods, and VMS methods. First of all, we obtain the optimality system. We then use FEM to obtain the discrete system. We obtain the theoretical stability results. We use the package freefem ++ to get the numerical results. We compare the stabilized solutions.

Keywords: Optimal control, Convection-diffusion equation, Stabilized fem

Introduction

Optimal control theory of viscous flows has several applications in science. There have been an interest in control problems of viscous problems in recent decades. Various numerical methods have been devoted to the solutions of the optimal control problems. The numerical methods are based on the computation of the derivatives of the function to be minimized. A gradient descent type method is frequently used to solve the control problems numerically. One of the most popular method to find the optimality system is the Lagrange method.

In this study, optimal control problem with a projection based VMS approach applied to both the state equation, convection disffusion, and its adjoint equation. In this technique, we add the global stabilization first and then subtract its effect from the large scales which are defined through projections.

We first formulate the problem and then find the stabilized discrete optimal control problem in variational form. We also give some numerical results.

Problem Formulation

Optimal control problem of convection diffusion equation is formulated as the following:

$$\min J(y,u) = \frac{1}{2}\|y - y_d\|_\Omega^2 + \frac{\alpha}{2}\|u\|_\Omega^2$$

$$subject\ to \quad -\epsilon\Delta y + \vec{b}.\nabla y + ry = u \quad in\ \ \Omega$$

$$y = 0 \quad on\ \Sigma = \partial\Omega.$$

Here, Ω is a convex polygonal domain. y and u denote the state and control variables, respectively. $\epsilon > 0$ denotes a constant diffusion parameter and r is a reaction coefficient. b is the convection field and $\alpha > 0$ regularization parameter. y_d is the desired state.

Solution of Optimal Control Problem

There are two different approaches for the discretization of the optimal control problems: optimize-then-discretize and discretize-then-optimize. We follow here the function based approach optimize- then-discretize.

Variational Form

The following well-known functional vector spaces are considered to define a variational formulation. Let

$$Y := H_0^1(\Omega) = \{y \in H^1(\Omega)\ and\ y = 0\ \ on\ \Sigma\}$$
$$U := L^2(\Omega).$$

We denote the state space by Y and the control space by U. We also assume the classical coercivity condition for convection diffusion equations which states that there exists a constant β such that $r - \frac{1}{2}\nabla.\vec{b} \geq \beta > 0$.

One can easily obtain the variational form of the state equation as: Find $y \in Y$ and $u \in U$ satisfying

$$\epsilon(\nabla y, \nabla v) + \left(\overrightarrow{b}.\nabla y, v\right) = (u, v)\ \ \forall v \in Y$$

Optimize-then-discretize

We follow Lagrange method. We assume that (y, u) is the optimal solution. Then, there exists p^* such that

$$L'(y^*, u^*, p^*) = 0$$

where $L(y,u,p) = J(y,u) - (p, -\epsilon\Delta y + (b.)^{+}\nabla y + ry - u$

Optimality Conditions

Standard optimal control theory gives that there exists an adjoint p satisfying [1]

$$-\epsilon\Delta p - \nabla.(\vec{b}p) + rp = y - y_d\ \ \ in\ \ \Omega$$
$$p = 0\ \ \ on\ \Sigma$$

and the pair (u, p) satisfies $(\alpha u + p, w - u) \geq 0\ \forall w \in U$.

Weak form of Adjoint Equation

Find $p \in Y$ satisfying

$$\epsilon(\nabla p, \nabla w) - \left(\nabla.\left(\overrightarrow{b}p\right), w\right) + (r, w) = \left(y - y_d, w\right)\ \ \forall w \in Y$$

Weak Solution

$$\epsilon(\nabla y^h, \nabla v^h) + \left(\overrightarrow{b}.\nabla y^h + ry^h, v^h\right) = (u^h, v^h)\ \ \forall v^h \in Y$$

$$\epsilon(\nabla p^h, \nabla w^h) - \left(\nabla.\left(\overrightarrow{b}p^h\right), w^h\right) + (r, w^h) = \left(y^h - y_d, w^h\right)\ \ \forall w^h \in Y$$

FEM

- Domain is divided into triangles or rectangles which are called elements
- On each element, a basis function is defined
- An approximate value of the unknown function is defined in terms of bases functions

Often, the diffusion is very small in comparison with the convection or reaction. This causes that the distribution of the respective quantity comprises so-called layer FEM does not give accurate numerical solution for convection dominated problems. We need stabilization.

Stabilization Methods

The most used numerical stabilization techniques for flow problems are
- Streamline Upwind Galerkin(SUPG)
- Pressure Stabilization(PSPG) methods
- Large Eddy Simulation(LES) methods
- Variational Multiscale Methods(VMS

SUPG and VMS

SUPG: One of the most successful linear stabilizations is the streamline upwind Petrov-Galerkin (SUPG) finite element method which consistently introduces artificial diffusion along streamlines. It combines good stability properties with a high accuracy away from layers.

VMS: The global stabilization was added to overall system first and then its effects were subtracted from the larger flow scales which are defined explicitly through some projections. Thus, stabilization acts only on the smallest resolved scales for both state and adjoint equations. VMS is easy in theoretical analysis.

Stabilized problem with VMS

$$\min J(y^h, u^h) = \frac{1}{2}\|y^h - y_d\|_\Omega^2 + \frac{\alpha}{2}\|u^h\|_\Omega^2$$

$$\textit{subject to} \quad \epsilon(\nabla y^h, \nabla v^h) + \left(\vec{b}.\nabla y^h + r y^h, v^h\right) + \sigma((I - P_h)\nabla y^h, (I - P_h)\nabla v^h) = (u^h, v^h) \ \forall v^h \in Y$$

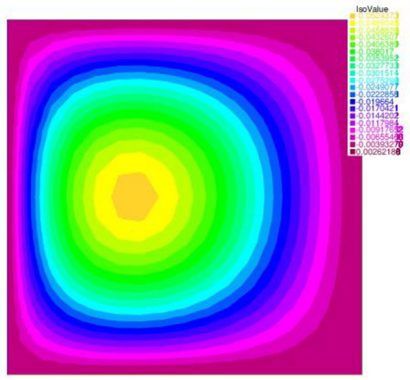
$$\epsilon(\nabla p^h, \nabla w^h) - \left(\nabla.\left(\vec{b} p^h\right), w^h\right) + (r, w^h) + \sigma((I - P_h)\nabla p^h, (I - P_h)\nabla w^h) = \left(y^h - y_d, w^h\right) \ \forall w^h \in Y$$

Here, P_h is the L^2 orthogonal projection of ∇y^h or ∇p^h. I stands for the identity operator and σ is a non-negative user selected stabilization parameter depending on the mesh width h [2].

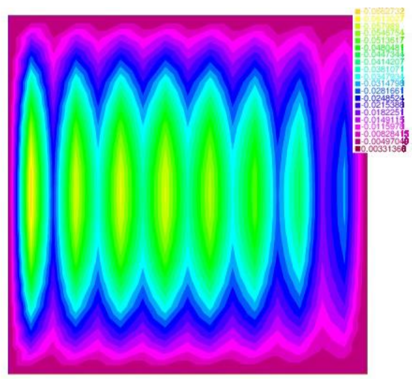
Numerical Example

We study in the domain $(0,1) \times (0,1)$ with a mesh resolution of 32×32. We choose the parameters as $\epsilon = 10^{-6}$, $\alpha = 0.001$, $b = \overrightarrow{(0,1)}$ r=0 and $y_d = \sin(\pi x)\sin(\pi y)$.

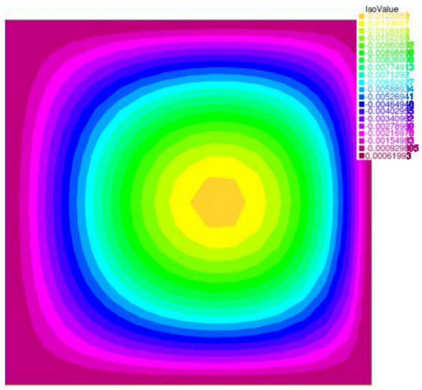
All computations are carried out with finite element software package Freefem++ [3].



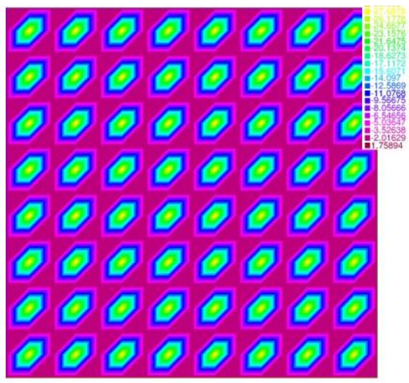
Stabilized optimal control



Unstabilized optimal control



Stabilized state



Unstabilized state

Conclusion

In this work, we have formulated optimal control problems of convection diffusion equations. After we have got the optimality conditions by using Lagrange approach, we apply FEM to find numerical solutions. We use a stabilization method for higher Reynolds numbers. We gave the numerical results to show the efficiency of the stabilization.

As a future work, we shall consider the control problems of the time dependent coupled flows.

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