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The Interaction of a Charged Particle in the Generalized Woods-Saxon Potential

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Abstract: An analytical solution of any given potential model presenting particle interaction is hot topic in physics. There are few potential models that can be analytically solved in literature. The analytically solvable potential models are the infinite and the finite well, the harmonic oscillator, the Coulomb and the Kratzer potential for any angular momentum quantum number. In this study, we examine the interaction of charged particle in the generalized Woods-Saxon Potential with an approximation to the effective potential by using the Hypergeometric function with physical boundary conditions and continuity requirement of the wave function. We obtain the bound state energy eigenvalues and corresponding wavefunction in closed form and discuss the effect of the potential parameters on the energy eigenvalues and corresponding eigenfunctions.

Keywords: Analytical solutions, Generalized Woods-Saxon potential, Charged particles

Introduction

Analytical solution of the generalized Woods-Saxon potential for the Schrödinger equation has a very important place in determining interaction of the particle in case of the bound, quasi-bound and scattering states (Zaichenko & Ol'Khovskii 1976; You et. al., 2002; Guo & Sheng, 2005; Panella et. al., 2010; Alpdoğan et. al., 2013; Bayrak&Aciksoz, 2015; Lütfüoğlu et. al., 2016; Lütfüoğlu 2018; Lütfüoğlu et. al. 2018). The generalized Woods-Saxon potential is given by (Satchler 1983),

$$V_N(r) = -\frac{V_0}{1+exp(\frac{r-R}{a})} + a\frac{d}{dr}(\frac{W_0}{1+exp(\frac{r-R}{a})}) \qquad (1)$$

where V_0 and W_0 are depths of the potential. R and a are the nuclear radius and surface thickness of the nucleus. The second term in Equation (1) is the surface term of the Woods-Saxon potential and crucial role for surface interaction of the nuclei (Satchler, 1983). In Figure 1, we plot the effective potential constituted the nuclear potential, Coulomb and centrifugal barrier potentials. It is seen that the surface potential in Equation (1) changes shape of the effective potential at the surface region. Depending on the sign of W_0 parameter, the nuclear potential is repulsive or attractive at the surface region.

There no an analytical solution of the generalized Woods-Saxon potential for the case of charged particles. We need to use an approximation for the Coulomb and centrifugal potentials. In literature, there are some approximation scheme. If we use Taylor expansion for the centrifugal potential and suggest an exponential potential as well as equating both of them, we obtained the parameter of the suggested exponential potential. This approximation firstly used by Pekeris for the analysis of the Rotation-Vibration Coupling in Diatomic Molecules (Pekeris, 1934). The second approximation for the Coulomb and Centrifugal potentials is called Greene-Aldrich approximation (Greene&Aldrich, 1976). Greene-Aldrich approximation is given by Equation (3). This approximation is convenient for small δ parameter. We use this approximation for the Coulomb and Centrifugal potentials in our calculations. Another approximation is suggested by Jia et. al. in the obtaining bound state energy eigenvalues and corresponding wave function of the Hulthen potential (Jia et. al., 2008).

In next section, we present method and calculation procedure. The brief discussion for obtained numerical results are given in Results and Discussion section. Then, we give a conclusion.

Method

In order to describe behavior of a proton orbiting around ^{56}Fe nucleus, we have to determine the effective potential between interacting nuclei. By transforming the Schrödinger equation with $R(r) = \dfrac{\chi(r)}{r}$ we obtain,

$$\left[\frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}(E - V_{eff}(r))\right]\chi(r) = 0, \quad V_{eff}(r) = V_N(r) - V_C(r) - \frac{l(l+1)\hbar^2}{2\mu r^2} \tag{2}$$

where $V_N(r)$ and $V_C(r)$ are the nuclear and Coulomb potentials. The last term in Equation (2) is the centrifugal potential. The reduced mass of interacting particle is $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$. Here, m_1 and m_2 are atomic mass of interacting particles. In the calculation we use point charge interaction between projectile and target nuclei. The Coulomb potential is $V_C(r) = \dfrac{Z_1 Z_2 e^2}{r}$. Here, Z_1 and Z_2 are proton numbers of the projectile proton and target ^{56}Fe. The effective potential is shown in Figure 1.
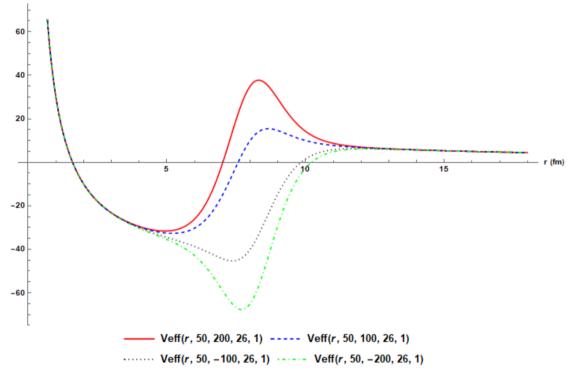

Figure 1. The effective potentials in Equation (1) which are the superposition of the nuclear, Coulomb and centrifugal barrier potentials for several nuclear parameters. The parameters in the effective potential can be given by the function as $V_{eff}(r, V_0, W_0, Z_2, l)$. We take $Z_1=1$, $R = 8$, $a = 0.65$

There is no analytical solution of the generalize Woods-Saxon potential with Coulomb potential for any angular momentum quantum number. We use an approximation for the Coulomb potential and centrifugal potential in Equation (2). The approximation is (Greene&Aldrich, 1976),

$$\frac{1}{r^2} = \frac{\delta^2 e^{-2\delta r}}{(1 + e^{-\delta r})^2} \quad \text{and} \quad \frac{1}{r} = \frac{\delta e^{-\delta r}}{1 + e^{-\delta r}} \tag{3}$$

where δ is screening parameter. Transforming Equation (2) with $x = \dfrac{1}{1+e^{\frac{r-R}{a}}}$ we can easily obtain,

$$\chi''(x) + \frac{1-2x}{x(1-x)}\chi'(x) + \frac{1}{x^2(1-x)^2}\left[-\varepsilon^2 + \alpha^2 x + \beta^2 x(1-x) - \gamma^2 x - \Lambda^2 x^2\right]\chi(x) = 0 \tag{4}$$

where,

$$-\varepsilon^2 = \frac{2\mu}{\hbar^2}a^2 E_n,\ \beta^2 = -\frac{2\mu}{\hbar^2}a^2 W_0,\ \alpha^2 = \frac{2\mu}{\hbar^2}a^2 V_0,\ \gamma^2 = \frac{2\mu}{\hbar^2}a^2 Z_1 Z_2 e^2 \delta,\ \Lambda^2 = a^2 \delta^2 L(L+1) \tag{5}$$

Investigating the singular points of Equation (4), we purpose an asymptotic wave function as $\chi(x) = x^\varepsilon (1-x)^\mu f(x)$ with $\mu^2 = \varepsilon^2 - \alpha^2 + \gamma^2 + \Lambda^2$ and insert into Equation (4). We obtain a Hypergeometric differential equation. Comprising between obtained differential equation with the Hypergeometric differential equation we can obtain the wave function as,

$$\chi(x) = x^\varepsilon (1-x)^\mu \, {}_2F_1\left(\frac{1}{2}\left(1+2\varepsilon+2\mu\pm\sqrt{1+4\beta^2}\right), \frac{1}{2}\left(1+2\varepsilon+2\mu\pm\sqrt{1+4\beta^2}\right); 1+2\varepsilon; x\right) \tag{6}$$

The wave function should satisfy the boundary conditions, namely, $\chi(r \to 0, x \to 1) = 0$. Therefore, the wave function explicitly analysis near the $x \cong 1$ by using following relation (Flügge, 1994),

$$\begin{aligned}
{}_2F_1(a,b;c;y) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\, {}_2F_1(a,b;a+b-c+1;1-y) + \\
&\quad \frac{\Gamma(c)\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)}(1-y)^{c-a-b}\, {}_2F_1(c-a,c-b;c-a-b+1;1-y)
\end{aligned} \tag{7}$$

As a result, the energy eigenvalue equation is obtained

$$\frac{\Gamma(a_1+b_1-c_1)\Gamma(c_1-a_1)\Gamma(c_1-a_1)}{\Gamma(c_1-a_1-b_1)\Gamma(a_1)\Gamma(b_1)}\left(1+e^{R/a}\right)^{-2\mu} = -1 \tag{8}$$

where,

$$a_1 = \frac{1}{2}\left(1+2\varepsilon+2\mu\pm\sqrt{1+4\beta^2+4\Lambda^2}\right),\ b_1 = \frac{1}{2}\left(1+2\varepsilon+2\mu\mp\sqrt{1+4\beta^2+4\Lambda^2}\right)$$
$$c_1 = 1+2\varepsilon$$

This equation is called a "transcendental equation". The energy eigenvalues in Equation (8) are independent from function of r. Since the energy eigenvalues is not obtained increasing r with step h for the given boundary conditions, obtained energy eigenvalues in Equation (8) are analytical results. But the energy eigenvalues cannot be extracted form Equation (8), we should be used any root finding method such as Bisection, Newton-Raphson etc.

Results and Discussion

We calculate the bound state energy eigenvalues of proton-^{56}Fe system with Equation (8) which is obtained for the effective potential in Equation (2) by using the approximation in Equation (3) to the Coulomb and Centrifugal potentials. In calculation we take $R = 4$ fm, $a = 0.4$ fm, $\delta = 0.01\,fm$, $Z_1 = 1$, $Z_2 = 26$ and change other potential parameters. The variation of the energy eigenvalues with the potential parameters are calculated by using Bisection method which is a method to calculate roots of a function (Table 1).

Table 1. Variation of the bound state energy eigenvalues for several potential parameters. In calculation we take potential parameter as V_0=60 MeV, W_0=30 MeV, l=0 and n=0. We keep the other parameter constant by calculating energy eigenvalues E_{nl} as a function of a parameter

V_0 (MeV)	E_{00} (MeV)	W_0 (MeV)	E_{00} (MeV)	n	E_{n0} (MeV)
0	-	0	−48.35	0	−49.17
20	−12.75	20	−48.88	1	−21.67
40	−30.44	40	−49.48	2	-
60	−49.17	60	−50.15	3	-
80	−68.30	80	-50.91	3	-

When we increase depth of the potential V_0 by keeping other potential parameters constant, we see that the energy increases. While the depth of the surface term W_0 in the effective potential increases, the depth of the effective potential also increases. In this way, the energy eigenvalues increase for the ground state. We calculate exited state of ^{56}Fe nucleus by the constant potential parameters. We see that increasing quantum number n, the energy eigenvalues decrease and after n>2, the proton could be unbound or quasi-bound in last column in Table 1. This station could be calculated by changing the boundary conditions.

Conclusion

We investigate the energy eigenvalues and corresponding wave function of the generalized Woods-Saxon potential with taking into account of the Coulomb potential and centrifugal barrier potentials. There is no analytical solution of the generalized Woods-Saxon potential with Coulomb and/or Centrifugal potential. In order to overcome this issue, we use the approximation to the Coulomb and Centrifugal potentials and obtain the approximate analytical solution for the energy eigenvalues and corresponding wave function. An application we calculate the energy eigenvalues of the proton orbiting around ^{56}Fe nucleus. Then we quantitatively investigate how energy eigenvalues is related to the potential parameters. This potential model could be important in order to investigate nuclear structure of the nuclei.

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References

Alpdoğan, S., Aydoğdu, O., Havare, A., (2013). Relativistic spinless particles in the generalized asymmetric Woods–Saxon potential. *J. Phys. A: Math. Theor., 46*:015301.

Bayrak, O., Aciksoz, E., (2015). Corrected analytical solution of the generalized Woods–Saxon potential for arbitrary states. *Phys. Scripta, 95*:015302.

Greene, R. L., Aldrich, C., (1976). Variational wave functions for a screened Coulomb potential. *Phys. Rev. A, 14*:2363.

Guo, J. Y. and Sheng, Z.Q., (2005). Solution of the Dirac equation for the Woods-Saxon potential with spin and pseudospin symmetry. *Phys. Lett. A, 338*(2): 90-96.

Jia, C.-S., Liu, J-Y., Wang, P.-Quan., (2008). A new approximation scheme for the centrifugal term and the Hulthén potential. *Physics Letters A 372*:(27-30) 4779.

Lütfüoğlu, B. C. (2018). An investigation of the bound-state solutions of the Klein-Gordon equation for the generalized Woods-Saxon potential in symmetry and pseudo-spin symmetry limits. *Eur. Phys. J. Plus, 133*(8): 309.

Lütfüoğlu, B. C., Akdeniz, F., Bayrak, O., (2016). Scattering, bound and quasi-bound states of the generalized symmetric Woods-Saxon potential. *J. of Math. Phys., 57*(3): 032103.

Lütfüoğlu, B. C., Lipovsky, J. and Kriz, J., (2018). Scattering of Klein-Gordon particles in the background of mixed scalar-vector generalized symmetric Woods-Saxon potential. *Eur. Phys. J. plus, 133*(1): 17.

Panella, O., Biondini, S. and Arda, A. 2010. New exact solution of the one-dimensional Dirac equation for the Woods-Saxon potential within the effective mass case. *J. Phys. A: Math. Theor.,43*(32): 325302.

Pekeris, C. L., (1934). The Rotation-Vibration Coupling in Diatomic Molecules. *Phys. Rev. 45*:98

Satchler, G. R. (1983). *Direct Nuclear Reactions.* Oxford, Clarendon Press.

You, G. J., Zheng, F. X., Xin, X. F., (2002). solution of the relativistic Dirac-Woods-Saxon problem. Phys. Rev. A, 66(6): 062105.

Zaichenko, A. K. and Ol'Khovskii, V. S., (1976). Analytic solutions of the scattering by potentials of the Eckart class. *Theo. And Math. Phys., 27*(2): 475-477.

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