The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM)
ISSN: 2602-3199

The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM), 2022

Volume 20, Pages 149-154

ICBAST 2022: International Conference on Basic Sciences and Technology

On Some Convergence Results for Hybrid Maximal Functions in Cat(0) Spaces

Eriola SILA
University of Tirana

Dazio PRIFTI
University of Tirana

Abstract: The study of CAT(0) spaces plays a crucial role not only in Reimann Geometry but also in Functional Analysis, Numerical Analysis, ect. The applications of Fixed Point Theory under some conditions in CAT(0) spaces have been on focus of many research papers. Their goals are in generalizing the expansive or contractive function, or giving new iteration which improves the convergent conditions for a function to its fixed points. The goal of this paper is the study of some convergence aspects of modified hybrid maximal functions in a complete CAT(0) space. There are given some sufficient conditions which guarantee the convergence of a sequence based on a new iteration for modified hybrid maximal function or hybrid partial maximal function using Fixed Point Theory.

Keywords: CAT(0) space, Iterative sequence, Fixed point, Modified hybrid maximal function

Introduction

The CAT(0) space concept was introduced for the first time by Alexandrov as geodesic spaces with zero curvature. Gromov in 1987 (Gromov, 1987) studied CAT(0) spaces as geodesic spaces where every geodesic triangle has a comparison tringle in Euclidian plane E^2. They satisfy the property that the geodesic distance for every two points in the geodesic triangle is smaller or equal to Euclidian distance of reciprocal points in Euclidian plane. Kirk (Kirk, 2003), (Kirk, 2004) proved some fixed point results in CAT(0) space. Many authors have published on this field. The study of contractive functions in these spaces has given a new theory which is applicated in Numeric Analysis. Mann in 1953 (Mann, 1953) has studied for the first time the convergence of a sequence based on an iteration in Banach spaces. Many researchers have studied the convergence of an iterative sequence in various spaces (Ishikawa, 1974), (Ullah et al., 2018), (Uddin et al., 2018). In this paper, there are given some convergence results for modified hybrid maximal functions in a complete CAT(0) space. A sequence is defined based on a new iteration. There is proved that this sequence converges to a fixed point of the modified hybrid maximal function or modified hybrid partial maximal function.

Preliminaries

Definition 1 (Bridson & Haefliger, 1999) Let (X,d) be metric space and $x, y \in X$. The map $\gamma: [0,l] \to X$ is called a geodesic curve if it completes the following conditions:

1. $\gamma(0) = x$;
2. $\gamma(l) = y$;
3. $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$, for every $t_1, t_2 \in [0,l]$.

4.

The image of γ is called geodesic segment that joins the point x, y.

The couple (X, d) is called (unique) geodesic metric space if for every two points in X there exists a (unique) geodesic curve that joins them.

It is denoted $\gamma(t0 + (1-t)l) = tx \oplus (1-t)y, t \in (0,1)$.

A subset $Y \subseteq X$ is called convex if for every geodesic segment that joins two points is included in Y.

Definition 2 (Bridson & Haefliger, 1999) Let (X, d) be a geodesic space. A geodesic triangle consits of three points $x_1, x_2, x_3 \in X$ and three geodesic segments. It is denoted $\Delta(x_1, x_2, x_3)$.

Definition 3 (Bridson & Haefliger, 1999) Let (X, d) be a geodesic space and $\Delta(x_1, x_2, x_3)$ be a geodesic triangle. A comparison triangle for geodesic triangle $\Delta(x_1, x_2, x_3)$ is the triangle $\bar{\Delta}(x_1, x_2, x_3) \coloneqq \Delta(\overline{x_1}, \bar{x}_2, \bar{x}_3)$ in Euclidian plane E^2 such that $d_{E^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$, for $i, j \in \{1,2,3\}$.

Definition 4 (Bridson & Haefliger, 1999) The geodesic space (X, d) is called CAT(0) space if for every triangle $\Delta(x_1, x_2, x_3)$ and $x, y \in \Delta$, the inequality $d(x, y) \le d_{E^2}(\bar{x}, \bar{y})$, for $\bar{x}, \bar{y} \in \bar{\Delta}$, holds.

Proposition 5 (Nanjaras, 2010) Let (X, d) be a CAT(0) space.

Then
$d((1-t)x \oplus ty, z) \le (1-t)d(x, z) + td(y, z)$
where $t \in [0,1]$, $x, y, z \in X$.

Proposition 6 (Dhompongsa, 2008) Let (X, d) be a CA/T(0) space. Then the following inequality is true

$$d^2((1-t)x \oplus ty, z) \le (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y)$$

for every $t \in [0,1], x, y, z \in X$.

In 1953 Mann (Mann, 1953) presented the iteration $y_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n T y_n$ where $T: E \to E$ and E is a Banach space.

Schu (Schu, 1991) modified this Mann's iteration by defining the sequence as follows:

$$y_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n T^n y_n$$

Definition 7 The function $T: X \to X$, where (X, d) is a CAT(0) space is called modified hybrid maximal if for each $x, y \in X$ and $0 < h < 1$ it satisfies the following condition:

$$d^2(T^p x, T^p y) \le h\max\{d^2(x, y), d^2(T^p x, x), d^2(y, T^p y), \frac{d^2(T^p x, y) + d^2(T^p y, x)}{4}\}.$$

Definition 8 The function $T: X \to X$, where (X, d) is a CAT(0) space is called modified hybrid partial maximal if for each $x, y \in X$ and $0 < h < 1$ it satisfies the following condition:

$$d^2(T^p x, T^p y) \le h\max\{d^2(x, y), \frac{d^2(T^p x, x) + d^2(y, T^p y)}{2}, \frac{d^2(T^p x, y) + d^2(T^p y, x)}{4}\}$$

Main results

The following result show a convergence result for a generalized hybrid maximal function.

Theorem 1 Let (X, d) be a complete CAT(0) space and K a nonempty, closed convex subset of X and $T: K \to K$ be a function that satisfies the condition:

$$d^2(T^p x, T^p y) \le \varphi(\max\{d^2(x,y), d^2(T^p x, y), d^2(x, T^p y)\frac{d^2(T^p x, x) + d^2(T^p y, y)}{2}\}$$

For $x, y \in K$, p is a natural number, $\varphi: [0, +\infty) \to [0, +\infty)$ is a continuous function such that $\varphi(0) = 0, \varphi(t) < t$ for each $t \in (0, +\infty)$. Suppose that the set of fixed points of the given mapping T is nonempty. Let y_0 be a point in K and the sequence $\{y_n\}_{n\in N}$ a sequence defined by the iteration $y_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n T^p y_n$, for every $n \ge 1$, and the sequence $\{\alpha_n\} \in [0, \frac{1}{2}]$ is convergent. Then $\lim_{n\to+\infty} d(y_n, x) = 0$, where x is the unique fixed point of T.

Proof. Since of fixed points of the function T is nonempty**,** there exists a point $x \in K$ such that $Tx = x$

Now we see $d^2(y_{n+1}, x) = d^2\big((1-\alpha_n)y_n \oplus \alpha_n T^p y_n, x\big)$

$$\le (1-\alpha_n)d^2(y_n, x) + \alpha_n \varphi(\max\left\{d^2(y_n, x), d^2(y_{n+1}, y_n), d^2(T^p x, x), \frac{d^2(y_{n+1}, x) + d^2(x, y_n)}{4}\right\})$$
$$- \alpha_n(1-\alpha_n)d^2(y_{n+1}, y_n)$$
$$< (1-\alpha_n)d^2(y_n, x) + \alpha_n \max\left\{d^2(y_n, x), d^2(y_{n+1}, y_n), \frac{d^2(y_{n+1}, x) + d^2(x, y_n)}{4}\right\}$$
$$- \alpha_n(1-\alpha_n)d^2(y_{n+1}, y_n)$$

Now we discuss the following cases:

If $\max\left\{d^2(y_n, x), d^2(y_{n+1}, y_n), \frac{d^2(y_{n+1}, x) + d^2(x, y_n)}{4}\right\} = d^2(y_n, x)$, we have

$$d^2(y_{n+1}, x) < (1-\alpha_n)d^2(y_n, x) + \alpha_n d^2(y_n, x) - \alpha_n(1-\alpha_n)d^2(y_{n+1}, y_n)$$
$$< (1-\alpha_n)d^2(y_n, x) + \alpha_n d^2(y_n, x) - \alpha_n(1-\alpha_n)d^2(y_{n+1}, x) - \alpha_n(1-\alpha_n)d^2(y_n, x)$$

Furthermore,

$\big(1 + \alpha_n(1-\alpha_n)\big)d^2(y_{n+1}, x) < (1 - \alpha_n(1-\alpha_n))d^2(y_n, x)$ and

$d^2(y_{n+1}, x) < \frac{(1-\alpha_n(1-\alpha_n))}{(1+\alpha_n(1-\alpha_n))}d^2(y_n, x) < d^2(y_n, x)$ for every $n \in N$.

If $\max\left\{d^2(y_n, x), d^2(y_{n+1}, y_n), \frac{d^2(y_{n+1}, x) + d^2(x, y_n)}{4}\right\} = d^2(y_{n+1}, y_n)$, we have

$$d^2(y_{n+1}, x) < (1-\alpha_n)d^2(y_n, x) + \alpha_n d^2(y_{n+1}, y_n) - \alpha_n(1-\alpha_n)d^2(y_{n+1}, y_n)$$
$$< (1-\alpha_n)d^2(y_n, x) + {\alpha_n}^2 d^2(y_{n+1}, y_n)$$
$$< (1-\alpha_n)d^2(y_n, x) + {\alpha_n}^2 d^2(y_{n+1}, x) + {\alpha_n}^2 d^2(y_n, x)$$

In addition,

$(1-{\alpha_n}^2)d^2(y_{n+1}, x) < (1 - \alpha_n + {\alpha_n}^2)d^2(y_n, x)$ and

$d^2(y_{n+1}, x) < \frac{(1-\alpha_n+{\alpha_n}^2)}{((1-{\alpha_n}^2))}d^2(y_n, x) < d^2(y_n, x)$ for every $n \in N$ since $\alpha_n \in [0, \frac{1}{2}]$.

If $\max\left\{d^2(y_n, x), d^2(y_{n+1}, y_n), \frac{d^2(y_{n+1}, x) + d^2(x, y_n)}{4}\right\} = \frac{d^2(y_{n+1}, x) + d^2(x, y_n)}{4}$, we have

$$d^2(y_{n+1}, x) < (1-\alpha_n)d^2(y_n, x) + \alpha_n \frac{d^2(y_{n+1}, x) + d^2(x, y_n)}{4} - \alpha_n(1-\alpha_n)d^2(y_{n+1}, y_n)$$
$$< (1-\alpha_n)d^2(y_n, x) + {\alpha_n}^2 d^2(y_{n+1}, y_n)$$
$$< (1-\alpha_n)d^2(y_n, x) + \alpha_n \frac{d^2(y_{n+1}, x) + d^2(x, y_n)}{4} - \alpha_n(1-\alpha_n)d^2(y_{n+1}, x)$$
$$- \alpha_n(1-\alpha_n)d^2(y_n, x)$$

From this inequality, we have

$$(1 + \frac{3\alpha_n}{4} - {\alpha_n}^2)d^2(y_{n+1}, x) < (1 - \frac{7\alpha_n}{4} + {\alpha_n}^2)d^2(y_{n+1}, x)$$

and, $d^2(y_{n+1},x) < \frac{(1-\frac{7\alpha_n}{4}+\alpha_n{}^2)}{(1+\frac{3\alpha_n}{4}-\alpha_n{}^2)} d^2(y_n,x) < d^2(y_n,x)$ for every $n \in N$ since $\alpha_n \in [0,\frac{1}{2}]$.

Consequently, we have proved that the sequence $\{d^2(y_n,x)\}$ is decreasing and bounded from above from zero. As a result, it converges to its infimum l. Suppose that $l \neq 0$.
So, $l \leq d^2(y_{n+1},x) <$

$$\leq (1-\alpha_n)d^2(y_n,x) + \alpha_n\varphi(\max\left\{d^2(y_n,x), d^2(y_{n+1},x) + d^2(x,y_{n+1}), \frac{d^2(y_{n+1},x) + d^2(x,y_n)}{4}\right\}) - \alpha_n(1-\alpha_n)(d^2(y_{n+1},x) + d^2(x,y_{n+1}))$$

Taking the limit when $n \to +\infty$, we have

$$l \leq (1-\alpha)l + \alpha\varphi(2l) - \alpha(1-\alpha)2l < (1-\alpha)l + 2l\alpha - 2\alpha(1-\alpha)l = (1-\alpha+2\alpha^2)l.$$

As a result, $l < (1-\alpha+2\alpha^2)l$ which is a contradiction because $\alpha \in [0,\frac{1}{2}]$. It yields that $l = 0$.

Theorem 2 Let (X,d) be a complete CAT(0) space and K a nonempty, closed convex subset of X and $T: K \to K$ be a function that satisfies the condition:

$$d^2(T^px, T^py) \leq \varphi(\max\left\{d^2(x,y), \frac{d^2(T^px,x) + d^2(T^py,y)}{2}, \frac{d^2(T^px,y) + d^2(T^py,x)}{4}\right\})$$

For $x,y \in K$, p is a natural number, $\varphi: [0,+\infty) \to [0,+\infty)$ is a continuous function such that $\varphi(0) = 0, \varphi(t) < t$ for each $t \in (0,+\infty)$. Suppose that the set of fixed points of the given mapping T is nonempty. Let y_0 be a point in K and the sequence $\{y_n\}_{n\in N}$ a sequence defined by the iteration $y_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n T^p y_n$, for every $n \geq 1$, and the sequence $\{\alpha_n\} \in [0,\frac{1}{2}]$ is convergent. Then $\lim_{n\to+\infty} d(y_n,x) = 0$, where x is the unique fixed point of T.

Proof:

We have that $d^2(y_{n+1},x) = d^2((1-\alpha_n)y_n \oplus \alpha_n T^p y_n, x)$
$\leq (1-\alpha_n)d^2(y_n,x) + \alpha_n\varphi(\max\left\{d^2(y_n,x), \frac{d^2(y_{n+1},y_n)+d^2(T^px,x)}{2}, \frac{d^2(y_{n+1},x)+d^2(x,y_n)}{4}\right\}) - \alpha_n(1-\alpha_n)d^2(y_{n+1},y_n) < (1-\alpha_n)d^2(y_n,x) + \alpha_n \max\left\{d^2(y_n,x), \frac{d^2(y_{n+1},y_n)}{2}, \frac{d^2(y_{n+1},x)+d^2(x,y_n)}{4}\right\} - \alpha_n(1-\alpha_n)d^2(y_{n+1},y_n)$

$$< (1-\alpha_n)d^2(y_n,x) + \alpha_n \max\left\{d^2(y_n,x), d^2(y_{n+1},y_n), \frac{d^2(y_{n+1},x) + d^2(x,y_n)}{4}\right\} - \alpha_n(1-\alpha_n)d^2(y_{n+1},y_n)$$

And we are in the same scheme of proof of Theorem 1. As a result, we have that $\lim_{n\to+\infty} d(y_n,x) = 0$.

Corollary 3 Let (X,d) be a complete CAT(0) space and K a nonempty, closed convex subset of X and $T: K \to K$ is a modified hybrid maximal function. Suppose that the set of fixed points of the mapping T is nonempty. Let y_0 be a point in K and the sequence $\{y_n\}_{n\in N}$ a sequence defined by the iteration $y_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n T^p y_n$, for every $n \geq 1$, and the sequence $\{\alpha_n\} \in [0,\frac{1}{2}]$ is convergent. Then $\lim_{n\to+\infty} d(y_n,x) = 0$, where x is the unique fixed point of T.

Proof: Taking $\varphi(t) = ht$, for $0 < h < 1$, we have that the hybrid maximal function satisfies the conditions of Theorem 1. Consequently, the result of Corollary 3 is immediately proved.

Corollary 4 Let (X,d) be a complete CAT(0) space and K a nonempty, closed convex subset of X and

$T: K \rightarrow K$ is a modified hybrid partial maximal function. Suppose that the set of fixed points of the mapping T is nonempty. Let y_0 be a point in K and the sequence $\{y_n\}_{n\in N}$ a sequence defined by the iteration $y_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n T^p y_n$, for every $n \geq 1$, and the sequence $\{\alpha_n\} \in [0, \frac{1}{2}]$ is convergent. Then $\lim_{n\rightarrow+\infty} d(y_n, x) = 0$, where x is a fixed point of T.

Conclusion

In this paper there are studied some convergence results for a sequence which is defined by a new iteration. This sequence converges to the fixed point of the respective function (given in Theorem 1, Theorem 2, Corollary 3, Corollary 4).

Recommendations

The authors suggest as further study the comparison between the convergence of Mann's iterative sequence and Schu's sequence with the one given in this paper to the fixed point of modified hybrid maximal functions and modified hybrid partial maximal functions respectively.

Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

Acknowledgements or Notes

* This article was presented as an oral presentation at the International Conference on Basic Sciences and Technology (www.icbast.net) held in Antalya/Turkey on November 16-19, 2022.

References

Bridson, M., Haefliger, A. (1999). *Metric spaces of non-positive curvature*. Springer-Verlag, Heilderberg

Dhompongsa. S, Panyanak. B. (2008). On -convergence theorems in CAT(0) space. *Comput Math. Appl 56*, 2572 -2579.

Gromov, M. (1987). Hyperbolic groups. *Essays in group theory, Math. Sci. Res. Inst. Publ*., 8, Springer, New York. 75–263.

Ishikawa, S. (1974). Fixed points by a new iteration method. *Proceedings of the American Mathematical Society*, *44*(1), 147-150.

Kirk, WA. (2003). Geodesic geometry and fixed point theory I. *Seminar of mathematical analysis, coleccion Abierta*. University of Seville, Secretary of Publications, Seville, Spain, *64*, 125-225.

Kirk. WA. (2004). Geodesic geometry and fixed point theory II. *International Conference on Fixed Point Theory and Applications*, (pp 113- 142). Yokohama Publishers, Yokohama.

Mann, WR. (1953). Mean value methods in iteration. *Proc AM. Math Soc*. *4,* 506-510.

Nanjaras, B., Panyanak, B., & Phuengrattana, W. (2010). Fixed point theorems and convergence theorems for Suzuki- generalized nonexpansive mappings in CAT(0) spaces. Nonlinear Analysis: Hybrid Systems, 4, 25–31.

Schu, J. (1991). Iterative construction of fixed points of asymptotically nonexpansive mappings. *J. Math. Anal. Appli.*, *158*, 407–413.

Uddin, I., Garodia, C. & Nieto, J.J. (2018). Mann iteration for monotone nonexpansive mappings in ordered CAT(0). space with an application to integral equations. *J Inequal Appl*, 339.

Ullah, K., Iqbal, K., & Arshad, M. (2018). *Some convergence results using K iteration process in CAT(0) spaces*. . Fixed Point Theory App 2018.

Author Information

Eriola Sila
Faculty of Natural Science, University of Tirana
Bulevardi 'Zogu i pare', Tirana, Albania
Contact E-mail: *eriola.sila@fshn.edu.al*

Dazio Prifti
Faculty of Natural Science, University of Tirana
Bulevardi 'Zogu i pare', Tirana, Albania

To cite this article:

Sila, E. & Prifti, D (2022). On some convergence results for hybrid maximal functions in CAT(0) spaces. *The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM), 20*, 149-154.