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The Construction of the (2+1)-Dimensional Integrable Fokas-Lenells Equation and its Bilinear form by Hirota Method

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Abstract: Integrable nonlinear differential equations are an important class of nonlinear wave equations that admit exact soliton of the solutions. In order to construct such equations tend to apply the method of mathematical physics, the inverse scattering problem method (ISPM), which was discovered in 1967 by Gardner, Green, Kruskal, and Miura. This method allows to solve more complicated problems. One of these equations is the (1+1)-dimensional integrable Fokas-Lenells equation, which was obtained by the bi-Hamiltonian method and appears as an integrable generalization of the nonlinear Schrödinger equation. In this paper, we have examined the (1+1)-dimensional Fokas-Lenells equation and in order to find more interesting solutions we have constructed the (2+1)-dimensional integrable Fokas-Lenells equation, whose integrability are ensured by the existence of the Lax representation for it. In addition, using the Hirota's method a bilinear form of the equation is constructed which was found by us, through which can be find its exact multisoliton solutions and build their graphs.

Keywords: Integrability, Lax representation, Fokas-Lenells equation, Hirota bilinear method, Soliton solutions

Introduction

Most physical laws are expressed in the form of differential equations, among them it is known that linear differential equations have the significant role. Besides them, some other physical phenomena in the surrounding world are nonlinear and require nonlinear equations for their description. Thereby, current days, an interest of the study of the precise nonlinear differential equations have been increased, which occur in all fundamental physical theories. At present, it is known that approximately two dozen one-dimensional nonlinear evolutionary equations integrated with the help of ISPM [1]-[2]. Multidimensional integrable systemswhich are containing derivatives with more than two variablesare particularly interesting. This led to the discovery of a whole interesting series of (2+1)-dimensional nonlinear equations integrable by the different approaches of the ISPM. In this paper, we have constructed the (2+1)-dimensional nonlinear Fokas-Lenells equation (FL).

So, to begin with, consider the (1+1)-dimensional FL equation. The FL equation was obtained by the bi-Hamiltonian method, which was proposed in 1995 by A.S. Fokas. Its the Lax representation was obtained in 2009 by A.S. Fokas and J. Lenells [3]-[4].

The investigated FL equation is as follows:

$$iq_{xt} - iq_{xx} + 2q_x - |q|^2 q_x + iq = 0, \tag{1}$$

where q is the complex shell of the field, the indices$x$and t denote partial derivatives with respect to the arguments x and t, and i - is the imaginary unit.

In addition, equation (1) can be rewritten in a modified form, with $r = -q^*$ (the sign "*" means complex conjugation) in the form

$$iq_{xt} - iq_{xx} + 2q_x - q_x qr + iq = 0, \tag{2a}$$
$$ir_{xt} - ir_{xx} - 2r_x + r_x rq + ir = 0. \tag{2b}$$

The Lax representation of the system of equations (2) has the form

$$\Psi_x = U(x,t,\lambda)\Psi, \tag{3a}$$
$$\Psi_t = V(x,t,\lambda)\Psi, \tag{3b}$$

where $\Psi(\lambda)$ is called an eigenfunction associated with λ, which is an isospectral parameter, and matrix operators U and V look like:

$$U(\lambda) = -i\lambda^2\sigma_3 + \lambda Q,$$
$$V(\lambda) = -i\lambda^2\sigma_3 + \lambda Q + V_0 + \frac{1}{\lambda}V_{-1} - \frac{i}{4\lambda^2}\sigma_3,$$

Here

$$Q = \begin{pmatrix} 0 & q_x \\ r_x & 0 \end{pmatrix}, \quad V_0 = i\sigma_3 - \frac{iqr}{2}\sigma_3, \quad V_{-1} = \frac{i}{2}\begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For equation (3), the following compatibility condition holds:

$$U_t - V_x + [U,V] = 0,$$

where $[U,V]$ - commutators.

(2+1)-dimensional FL equation

Now we turn to the construction of the (2+1)-dimensional FL equation. To do this, we define the Lax representation of the required equation in the following form (below, its explicit form will be found) [5] - [6]:

$$\Psi_x = U(x,t,\lambda)\Psi, \tag{4a}$$
$$\Psi_t = \alpha\Psi_y + W(x,t,\lambda)\Psi, \tag{4b}$$

where α - constant, matrix operators U and W take as

$$U = -i\lambda^2\sigma_3 + \lambda Q,$$
$$W = W_0 + \frac{1}{\lambda}W_{-1} + \frac{1}{\lambda^2}W_{-2},$$

here W_0, W_{-1} and W_{-2} - unknown 2×2 matrices that we need to define

$$W_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad W_{-1} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \qquad W_{-2} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

where a_{ij}, b_{ij} and c_{ij} - constants.

From the cross differentiation of equation (4), we find the following compatibility condition for the (2+1)-dimensional equation of the FL:

$$U_t - W_x + [U,W] - \alpha U_y = 0. \tag{5}$$

From equations (5) it obtains

$$\lambda Q_t - \alpha\lambda Q_y - W_{0x} - \frac{1}{\lambda}W_{-1x} - \frac{1}{\lambda^2}W_{-2x} - i\lambda^2[\sigma_3, W_0] - i\lambda[\sigma_3, W_{-1}] - i[\sigma_3, W_{-2}] +$$
$$+ \lambda[Q, W_0] + [Q, W_{-1}] + \frac{1}{\lambda}[Q, W_{-2}] = 0. \tag{6}$$

We have expanded equation (6) in powers of the isospectral parameter λ

$$\lambda^2 \quad : \quad -[\sigma_3, W_0] = 0, \tag{7}$$
$$\lambda^1 \quad : \quad Q_t - \alpha Q_y - i[\sigma_3, W_{-1}] + [Q, W_0], \tag{8}$$
$$\lambda^0 \quad : \quad -W_{0x} - i[\sigma_3, W_{-2}] + [Q, W_{-1}] = 0, \tag{9}$$
$$\lambda^{-1} \quad : \quad -W_{-1x} + [Q, W_{-2}] = 0, \tag{10}$$
$$\lambda^{-2} \quad : \quad -W_{-2x} = 0. \tag{11}$$

Equation (7) gives

$$a_{12} = 0, \qquad a_{21} = 0,$$

from equation (8) it follows that

$$q_{xt} - \alpha q_{xy} - 2ib_{12} - q_x a_{1122} = 0, \tag{12a}$$
$$r_{xt} - \alpha r_{xy} + 2ib_{21} + r_x a_{1122} = 0, \tag{12b}$$

from the equation (9) we obtain the following:

$$a_{11x} = q_x b_{21} - b_{12} r_x, \qquad a_{22x} = -q_x b_{21} + b_{12} r_x, \qquad a_{11x} = -a_{22x}, \tag{13}$$
$$c_{12} = \frac{iq_x b_{1122}}{2}, \qquad c_{21} = \frac{ir_x b_{1122}}{2},$$

equation (10) gives

$$b_{11x} = q_x c_{21} - c_{12} r_x, \qquad b_{22x} = r_x c_{12} - c_{21} q_x, \qquad b_{11x} = -b_{22x}, \tag{14}$$
$$b_{12} = -qc_{1122}, \qquad b_{21} = rc_{1122}, \tag{15}$$

and from equation (11) leaves

$$c_{11x} = c_{12x} = c_{21x} = c_{22x} = 0.$$

Now substituting equations (14)-(15) into equation (13), we obtain

$$a_{11} = qrc_{1122} + a_{110}, \qquad a_{22} = -qrc_{1122} + a_{220}.$$

Unknown constants are set as

$$a_{110} = i, \qquad a_{220} = -i, \qquad c_{21} = c_{12} = 0, \qquad c_{11} = -i/4, \qquad c_{22} = i/4,$$

then the matrix W_{-1} and W_0 matrix components takes the following values

$$b_{11} = b_{22} = 0, \qquad b_{12} = \frac{iq}{2}, \qquad b_{21} = -\frac{ir}{2},$$
$$a_{12} = a_{21} = 0, \qquad a_{11} = -\frac{iqr}{2} + i, \qquad a_{22} = \frac{iqr}{2} - i.$$

Thus, equations (12) take the form

$$q_{xt} - \alpha q_{xy} + q + iqrq_x - 2iq_x = 0, \tag{16a}$$
$$r_{xt} - \alpha r_{xy} + r - ir_x qr + 2ir_x = 0. \tag{16b}$$

Equation (16) is multiplied by i, then

$$iq_{xt} - i\alpha q_{xy} + iq - qrq_x + 2q_x = 0, \tag{17a}$$
$$ir_{xt} - i\alpha r_{xy} + ir + r_x qr - 2r_x = 0. \tag{17b}$$

Thus, setting the constant $\alpha = 1$ in equations (17), we finally obtain the (2+1)-dimensional Fokas-Lenells equation, which has the form

$$iq_{xt} - iq_{xy} + 2q_x - q_x qr + iq = 0, \tag{18a}$$
$$ir_{xt} - ir_{xy} - 2r_x + r_x rq + ir = 0. \tag{18b}$$

A Lax representation for the found (2+1)-dimensional FL equation (18) is as follows:

$$\Psi_x = U\Psi = (-i\lambda^2\sigma_3 + \lambda Q)\Psi,$$
$$\Psi_t = \Psi_y + W\Psi = \Psi_y + (W_0 + \frac{1}{\lambda}W_{-1} - \frac{i}{4\lambda^2}o_3)\Psi,$$

where

$$Q = \begin{pmatrix} 0 & q_x \\ r_x & 0 \end{pmatrix}, \qquad W_0 = i\sigma_3 - \frac{iqr}{2}\sigma_3, \qquad W_{-1} = \frac{i}{2}\begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The bilinear form of the (2+1)-dimensional FL equation

Some soliton solutions of integrable nonlinear differential equations can be found by Hirota bilinear method [7]. In order to do this, firstly, it needs to construct the bilinear form of the equation by this method and use the constructed bilinear form to find the solutions of the desired equation. In what follows we shall use this method to obtain multisoliton solutions for equation (18).

Thus, we turn to the construction of the bilinear form of the (2+1)-dimensional FL equation. To do this, we rewrite our system of equations (18), with $r = -q^*$. Then the system of equations (18) takes the following form:

$$iq_{xt} - iq_{xy} + 2q_x - q_x|q|^2 + iq = 0. \tag{19}$$

The bilinear form of equation (19) is established by the following statement:

$$q = \frac{g}{f}, \tag{20}$$

where g and f - complex functions from x, y and t.
Now, we find the derivatives of equation (20) alternately with respect to the arguments x, xt and xy

$$q_x = \frac{g_x f - gf_x}{f^2}, \tag{21}$$
$$q_{xt} = \frac{g_{xt} f - g_x f_t - g_t f_x + gf_{xt}}{f^2} - \frac{2g}{f^3}(f_{xt} f - f_x f_t), \tag{22}$$

$$q_{xy} = \frac{g_{xy} f - g_x f_y - g_y f_x + g f_{xy}}{f^2} - \frac{2g}{f^3}(f_{xy} f - f_x f_y). \tag{23}$$

Substituting equations (21)-(23) into equation (19) and applying the following properties of Hirota operators:

$$D_x^m D_t^n f \cdot f^* = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x,t) f^*(x',t')|_{x'=x,t'=t}, \tag{24}$$

where x', t' as two formal variables, $f(x,t)$ and $f^*(x',t')$ - two functions, m and n - two non-negative integers, we obtain

$$\frac{1}{f^2}\left(iD_x D_t(g \cdot f) - iD_x D_y(g \cdot f) + 2D_x(g \cdot f) + igf\right) - \frac{2g}{f^2 f^*}\left(iD_x D_t(f \cdot f^*) - iD_x D_y(f \cdot f^*) - \frac{1}{2}D_x(g \cdot g^*)\right) +$$

$$+\frac{2gf_x}{f^3 f^*}\left(iD_t(f \cdot f^*) - iD_y(f \cdot f^*) - \frac{1}{2} g \cdot g^*\right) - \frac{2g}{f^2 f^*}\left(iD_t(f \cdot f_x^*) - iD_y(f \cdot f_x^*) - \frac{1}{2} g \cdot g_x^*\right) = 0. \tag{25}$$

Thus, by separating equation (25) in powers f, we obtain the bilinear form of the (2+1)-dimensional Fokas-Lenells equation

$$iD_x D_t(g \cdot f) - iD_x D_y(g \cdot f) + 2D_x(g \cdot f) + igf = 0, \tag{26}$$

$$iD_x D_t(f \cdot f^*) - iD_x D_y(f \cdot f^*) - \frac{1}{2}D_x(g \cdot g^*) = 0, \tag{27}$$

$$iD_t(f \cdot f^*) - iD_y(f \cdot f^*) - \frac{1}{2} g \cdot g^* = 0, \tag{28}$$

$$iD_t(f \cdot f_x^*) - iD_y(f \cdot f_x^*) - \frac{1}{2} g \cdot g_x^* = 0, \tag{29}$$

where D_x, D_y and D_t - bilinear differential operators.

Conclusion

In this paper, we have examined the (1+1)-dimensional Fokas-Lenells equation and its Lax representation. By the method of cross differentiation, the components of the Lax representation for the (2+1)-dimensional FL equation have been reconstructed and, using some transformations, such as: Darboux, Backlund, and Hirota, can be found exact solutions of the found equation. In addition, we have obtained the bilinear form of equation (18), by using the bilinear form which was constructed by us, in our next papers we will try to find multisoliton solutions of the (2+1)-dimensional FL equation and their graphs will be constructed.

References

Абловиц М., Сигур Х. (1987). *Солитоны и метод обратной задачи.* М.: «Мир».

Захаров В.Е., Манаков С.В., Новиков С.П., Питаевский Л.П. (1980). *Введение в теорию интегрируемых систем*. М.: «Наука».

Jingsong H., Shuwei X., Kuppuswamy P. (2012). Rogue Waves of the Fokas-Lenells Equation. *Journal of the Physical Society of Japan*, *81*(12).

Lenells J., Fokas A.S. (2009). On a novel integrable generalization of the nonlinear Schrödinger equation. *Nonlinearity, 22*(1), 11-27.

Myrzakulov R., Mamyrbekova G., Nugmanova G., Lakshmanan M. (2015). Integrable (2+1)-dimensional spin models with self-consistent potentials. *Symmetry*, *7*(3), 1352-1375.

Myrzakul Akbota, Myrzakulov R. (2017). Integrable motion of two interacting curves, spin systems and the Manakov system. *International Journal of Geometric Methods in Modern Physics, 14*(7).

Борзых А.В. (2002). Метод Хироты и солитонные решения многомерного нелинейного уравнения Шредингера. *Сибирский математический журнал, 43*(2), 267-270.

Matsuno Y. (2012). A direct method of solution for the Fokas-Lenells derivative nonlinear Schrödinger equation: II. Dark soliton solutions. *Journal of Physics A: Mathematical and Theoretical, 45*(47).

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