The Eurasia Proceedings of
Educational & Social Sciences (EPESS)
ISSN: 2587-1730

**The Eurasia Proceedings of Educational & Social Sciences (EPESS), 2019**

**Volume 7, Pages 338-345**

**IConTES 2019: International Conference on Technology, Engineering and Science**

# Investigation of the Dynamic Systems' Phase Portraits

**Irina ANDREEVA**
Peter the Great St. Petersburg Polytechnic University

**Tatiana EFIMOVA**
Peter the Great St. Petersburg Polytechnic University

**Abstract**: Dynamic systems in a broad sense may be considered as mathematical models of processes and phenomena, where any statistical events we may disregard. The dynamic systems theory investigates curves, defined by differential equations. The same time the laws of Nature are written using the language of differential equations, as the great French mathematician and encyclopedist of the nineteenth and twentieth centuries J.H. Poincare has taught. Thus, these laws are written using dynamic systems. A study of a given family of dynamic systems depending on several parameters means splitting of a phase space belonging to the dynamic system under consideration into trajectories and investigation of the limit behavior of them with the aim to reveal and describe their positions of equilibrium, and to find the so-called sinks and sources. Also, very important are the question of the stability of equilibrium states and their types, as well as the question of a roughness of a system. Rough dynamic systems can preserve their qualitative character of motion under some considerably small changes in parameters of the system. The paper is devoted to the original investigation of a broad family of polynomial dynamic systems, depending on multiple parameters.

**Keywords:** Dynamic systems, Trajectories, Phase portraits, Equilibrium states, Differential equations, Poincare sphere, Poincare disk, Singular points, Separatrices, Limit cycles

## Introduction

The unique "scientific position" of a dynamic system is to serve as a mathematical model of some process, where any statistical events (fluctuations, for example) could be omitted and put out of investigator's consideration. Any dynamic system has the two main features, the two main characteristics of it. Namely those characteristics are: 1) the initial state of a dynamic system, and 2) a law of this system's transformation into some different state. A totality including all admissible states of a given dynamic system is called a *phase space* of it. There can be separated the two main categories of dynamic systems: one of them includes all dynamic systems with the discrete time (the so-called *cascades*), while the other category includes dynamic systems with the continuous time (the *flows*). For cascades their behavior is described with a sequence of different states. For flows their state is defined on an imaginary or on a real axis per each time moment. Flows and cascades represent the key fields of investigations in such branches of science as the topological dynamics and the symbolic dynamics.

But the same time the both abovementioned principally different types of dynamic systems usually can be described using a certain autonomous system of differential equations, which is defined in some domain and satisfies in this domain the conditions of the Cauchy theorem. The Cauchy theorem of existence and uniqueness of solutions of differential equations is the one of main theorems in the theory of differential equations. Under such an approach the singular points of differential equations will be matched to dynamic systems' equilibrium positions, while periodical solutions of differential equations will match to dynamic systems' closed phase curves.

The actual key problem of the dynamic systems' theory is an investigation of curves, which are defined with differential equations.

A process of such a study involves the following several stages:
1) we need to split a phase space of a taken dynamic system into trajectories and then
2) investigate their limit behavior.
That means:
2.1) find out and identify classification types of equilibrium positions,
2.2) reveal and investigate possible attracting and repulsive manifolds of a dynamic system under consideration.

The additional key (greatly important) notions introduced in the dynamic systems theory are the following notions:

1) a notion of a stability of equilibrium states of the system. Here we mean the following property of the dynamic system: its ability to remain near some equilibrium state - or on an indicated manifold - for a voluntary prolongated time period provided that small enough changes of initial data take place, as well as

2) a notion of a dynamic system's roughness (this property means an ability of saving of its characteristics in the face of small changing of the whole model itself). A rough dynamic system shows such a type of its behavior, that it preserves the (qualitative) character of motion in the face of a (satisfactory) small changes in the very system's parameters.

Modern specialists in the field of the theory of dynamic systems honor Jules Henri Poincare, the great French mathematician and encyclopedist (1854 – 1912), as the founder of the qualitative theory of differential equations. J. H. Poincare was one of the last encyclopedists – being able to keep in his mind almost all contemporary to him branches - not only of the pure mathematics, but also of physics, celestial mechanics, astronomy etc. For example, J.H. Poincare has laid the very foundation of the relativity theory.

J.H. Poincare has revealed, that each normal second-order autonomous differential system having polynomials in its right parts principally allows the total exhaustive qualitative investigation of it on an (extended) real plane $\bar{R}^2_{x,y}$ [1].

Several special different types of such polynomial dynamic systems were satisfactory studied since the J.H. Poincare times. Modern mathematicians have exhaustively investigated some categories of such systems, i.e. the quadratic polynomial dynamic systems [2], dynamic systems involving nonzero linear terms in their right parts, cubic homogeneous polynomial systems, and also polynomial dynamic systems including nonlinear homogeneous terms of some odd degrees (such as 3, 5, 7) [3], dynamic systems, which have a center or a focus in a singular point O (0,0) [4], and several additional different system's types.

In the present paper let us discuss some special broad family of dynamic systems on an arithmetical (real) plane of their phase variables *x, y*

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y), \tag{1}$$

where *X (x, y), Y (x, y)* be the reciprocal polynomial forms of *x* and *y*, *X* be a cubic, while *Y* be a square polynomial form, $X(0,1) > 0$, $Y(0, 1) > 0$.

The first of our established goals is to reveal and construct in a Poincare disk all possible types of phase portraits, which are appear to be admissible for the Eq. (1) – dynamic systems.

The second of our established goals is to determine and describe the (close to the coefficient ones) criteria of every such a phase portrait existence. The J.H. Poincare's sequential mapping method helps us to achieve our goal on this way: at first it is necessary to proceed the central mapping of an augmented with a line at infinity phase real plane $\bar{R}^2_{x,y}$ (from the center (0, 0, 1) of the Poincare sphere ∑) onto the sphere $\sum : X^2 + Y^2 + Z^2 = 1$ where the diametrically opposite points are considered to be identified, and after this step, at the second, we proceed the orthogonal mapping of the enclosed lower semi sphere of a Poincare sphere ∑ to the Poincare disk $\bar{\Omega}: x^2 + y^2 \leq 1$, where we consider to be identified the points of its boundary *Γ* which are diametrically opposite one to another.

As it was already mentioned above in the context, the disk $\bar{\Omega}$ and the sphere ∑ are further called correspondingly the Poincare disk and the Poincare sphere [1].

## Some Important Notations and Definitions

$\varphi(t,p)$, $p=(x,y)$ –a fixed point := a motion (a solution) of an Eq.(1) – system with initial data $(0,p)$.

$L_p : \varphi = \varphi(t,p), t \in I_{max}$, - a trajectory of a motion $\varphi(t,p)$.

$L_p^{+(-)}$ := +(-)- a semi trajectory of a trajectory $L_p$.

*O*-curve of a system := the system's semi trajectory $L^s_p$ ($p \neq O$, $s \in \{+,-\}$), which is adjoining to a point *O* with the condition $st \to +\infty$.

$O^{+(-)}$- curve of a system := this system's *O*-curve $L_p^{+(-)}$.

$O_{+(-)}$-curve of a system := this system's *O*-curve, which is adjoining to a point *O* from a domain $x > 0$ ($x < 0$).

*TO*-curve of a system := this system's *O*-curve, which, being supplemented by a point *O*, touches some ray in the point *O*.

A nodal bundle of *NO*-curves of a system := an open continuous family of this system's *TO*-curves $L^s_p$, where $s \in \{+,-\}$ is a fixed index, $p \in \Lambda$, $\Lambda$ - is a simple open arc, $L^s_p \cap \Lambda = \{p\}$.

A saddle bundle of *SO*-curves of a system, a separatrix of the point *O*:= is a fixed *TO* -curve, which is not included into any bundle of *NO*-curves of a system.

*E, H, P* : an elliptical, a hyperbolic, a parabolic *O*-sectors of a system.

A topological type (T-type) of a finite singular point *O* of a system := a word $A_O$ including letters *N, S* (a word $B_O$ including letters *E, H, P*), which is used to describe a circular order of nodal and saddle bundles *N, S* of this system's *O*-curves (of this system's elliptical, hyperbolic, parabolic *O*-sectors *E, H, P* ) while traversing the finite singular point *O* counterclockwise = in the «+»-direction, starting from some of them.

$$P(u) := X(1,u) \equiv p_0 + p_1 u + p_2 u^2 + p_3 u^3,$$
$$Q(u) := Y(1,u) \equiv a + bu + cu^2.$$

**Note A.** For each Eq.(1) - system:

1) The T-type of a finite singular point *O* in its form $B_O$ is easy to construct using the T-type of this singular point in the form $A_O$ , and backwords (we ought to know the both forms, see the Corollary 1 below);

2) the real roots of a polynomial *P*(*u*) (polynomial *Q*(*u*)) appear to be in fact the angular coefficients of the isoclines of infinity (the isoclines of a zero);

3) writing out the real roots of the system's special polynomials *P*(*u*), *Q*(*u*), separately or all together, we agree always to number the roots of each one of those polynomials in an ascending order.

## A Singular Point O (0, 0) and Its Topological Type

Now we need to:

1) reveal and enlist all the existing for these dynamic systems of Eq. (1) *O*-curves, and

2) to split the whole their totality into the nodal and saddle bundles *N, S*.

The method of exceptional directions of a dynamic system in the finite singular point *O* will be used in order to achieve this established goal [1]. The exceptional directions' equation for the point *O* of the Eq.(1) – dynamic systems family will be written in the form

$$xY(x,y) \equiv x(ax^2 + bxy + cy^2) = 0.$$

So the several different cases which are listed below may be realized for this equation:

1) $d \equiv b^2 - 4ac > 0$ - in this case the exceptional directions' equation defines the simle straight lines described with the equations $x = 0$ and $y = q_i x$, $i = 1,2$, $q_1 < q_2$,

2) $d = 0$ - in this case the exceptional directions' equation defines a straight line described by the equation $x = 0$ as well as a double straight line described with equations $y = qx, q = -\frac{b}{2c}$,

3) $d < 0$ – under this condition the exceptional directions' equation defines only one straight line $x = 0$ .

**Theorem A.** A topological type of a finite singular point *O (0, 0)* of a family of dynamic systems corresponding to the equations (1) is described with the so-called words $A_O$ and $B_O$

These words can obtain the following structures:

1) if $d > 0$ - depending on signs of values $P(q_i), i = 1,2,$ have forms, indicated in a Table 1,

2) if $d = 0$ - depending on signs of values $q$ and $P(q)$ - have forms, indicated in a Table 2,

3) if $d < 0$ a form: $A_O = S_0S^0$ , $B_O = HH$: [5].

**Table 1**. Topological type of a finite singular point *O(0, 0)* in the case when *d > 0*

| $r$ | $P(q_1)$ | $P(q_2)$ | $A_O$ | $B_O$ |
|---|---|---|---|---|
| 1, 4 | + | + | $S_0S_+^1N_+^2S^0N_-^1S_-^2 = S_0S_+^1NS_-^2$ | $PH^2$ |
| 2 | – | – | $S_0N_+^1S_+^2S^0S_-^1N_-^2 = NS_+^2S^0S_+^1$ | $PH^2$ |
| 3, 6 | – | + | $S_0N_+^1N_+^2S^0S_-^1S_-^2$ | $PEPH^3$ |
| 5 | + | – | $S_0S_+^1S_+^2S^0N_-^1N_-^2$ | $H^3PEP$ |

Table 2. Topological type of a finite singular point *O(0, 0)* in the case when *d* = 0

| $q$ | $P(q)$ | $A_O$ | $B_O$ |
|---|---|---|---|
| + | + | $S_0S_+N_+S^0$ | $H^2P$ |
| – | – | $S_0N_+S_+S^0$ | $PH^2$ |
| + | – | $S_0S^0S_-N_-$ | $H^2P$ |
| – | + | $S_0S^0N_-S_-$ | $PH^2$ |
| 0 | + | $S_0S_+NS_-$ | $H^2P$ |
| 0 | – | $NS_+S^0S_-$ | $PH^2$ |

**Note B**. Explanations and meanings of the new symbols, introduced in the Theorem A.

The symbol $S_0$ is used to indicate a saddle bundle *S*, which is adjoining to the finite singular point *O*(0, 0) from the domain $x > 0$ along the semi axis $x = 0, y < 0$, with the condition that $t \to +\infty$ .

Similarly, the symbol $S^0$ is used to indicate a saddle bundle *S*, which is adjoining to the finite singular point *O*(0, 0) from the domain $x > 0$ along the semi axis $x = 0, \ y > 0,$ with the condition that $t \to -\infty$.

A lower sign index «+» or «-» of each nodal or saddle bundle *N* or *S*, different from $S_0$ or $S^0$, is used to indicate will the bundle under consideration consist of $O_+$-curves or of $O_-$-curves. An upper index 1 or 2 of each bundle means: will the *O*-curves of this bundle be adjoining to the finite singular point *O* along a straight line defined with the equation $y = q_1x$ or, oppositely, along a straight line defined by the equation $y = q_2x$ .

In the Table 2 above, see lines with the numbers 5 and 6, the node bundle *N* has not any lower sign index, since it contains both $O_+$ -curves and $O_-$ -curves together.

**Corollary A.** As we can conclude from the Theorem A above, all the dynamic systems belonging to the broad family described with the Equations (1) have no limit cycles on the arithmetical plane of their phase variables $R^2_{x;y}$ .

Really, such an important feature of the dynamic system as the limit cycle principally could exist and surround a finite singular point *O (0,0)* of the Eq.(1) - system, and in this situation the Poincare index of that singular point will necessarily be equal to 1 [1]. But let us take into consideration the very important formula for calculation of the index of an isolated singular point of a smooth dynamic system, the formula of Bendixon, which looks like the follows:

$$I(O) = 1 + \frac{e-h}{2},$$

where *e* is a number of elliptical *O*-sectors of the dynamic system, and *h* is a number of its hyperbolic *O*-sectors. The Bendixon's formula together with the formulated above Theorem A mean, that for the finite isolated singular point *O (0, 0)* of any smooth dynamic system described with the Equations (1) the Poincare index *I*(*O*) = 0. So there are no limit cycles among the trajectories of these systems.

**Corollary B.** There exist 11 (eleven) different topological types for the isolated finite singular point *O (0, 0)* of the Eq.(1) – systems family. After the detailed thorough analysis of these eleven topological types we can conclude, that for each Eq. (1) – dynamic system there may exist no more than 4 (four) separatrices of the isolated finite singular point *O (0, 0).* The actual amount of such separatrices may vary from 2 to 4 ones.

## Investigation of the İnfinitely Remote Singular Points

Obviously a question of a great interest is the investigation of a behavior of trajectories of the dynamic systems under consideration in the neighborhood of infinity. Such a field of studies demands using of the sequential mappings method. The sequential mappings method is a powerful instrument in the qualitative theory of differential equations and dynamic systems. It was invented by Jules Henri Poincare and provides for the sequential implementation of the two Poincare transformations (famous enough among the researches in this field) [1].

The first Poincare transformation

$$x = \frac{1}{z}, \quad y = \frac{u}{z} \quad \left(u = \frac{y}{x}, \quad z = \frac{1}{x}\right)$$

maps a phase arithmetical plane $R^2_{x,y}$ of the dynamic systems described by the equations (1) anambiguosly onto the so-called Poincare sphere $\sum$: $x^2 + y^2 + z^2 = 1$ (where $z = -Z$ [1]).The diametrically opposite points on the Poincare sphere $\sum$ we consider to be identified. Firstly we consider the Poincare sphere $\sum$ without it`s equator *E*. But also the first Poincare transformation maps the infinitely remote straight line of a plane $\overline{R^2_{x,y}}$ onto the equator *E* of the Poincare sphere $\sum$. The diametrically opposite points on the equator *E* of the Poincare sphere $\sum$ we considered to be identified similarly.

The described above first Poincare transformation (and mapping) translates the dynamic system described by the equations (1) into the system, which in the new coordinates of the first Poincare transformation $u, z$ (and after a time change) has the form

$$\frac{du}{d\tau} = P(u)u - Q(u)z, \ \frac{dz}{d\tau} = P(u)z,$$

where $P(u) :\equiv X(1, u)$ and $Q(u) :\equiv Y(1, u)$ are reciprocal polynomials (this special term means that they would not have common roots and, consequently, would not have common multipliers in their decompositions into the forms of lower degrees).

The obtained new dynamic system will be considered and determined on the whole Poincare sphere $\sum$, including the equator *E* of it. Also this new system will be determined on the whole $(u, z) -$ plane $\alpha^*$, which is tangent to a sphere $\sum$ at a point C = (1, 0, 0). We need to investigate the new system on the plane $\overline{R^2_{u,z}}$, and it will be necessary to project the results received in this study onto a closed Poincare disk (or a Poincare circle in the other varient of this term) $\overline{\Omega}$, mapping firstly a plane $R^2_{u,z}$ onto the Poincare sphere $\sum$ from the center of the sphere $\sum$, and secondly the lower semi sphere $\overline{H}$ of the Poincare sphere $\sum$ onto the Poincare disk $\overline{\Omega}$, that means – (via the orthogonal mapping) onto a closed unit disk of a plane $R^2_{x,y}$.

For the appeared as a result of the described above first Poincare mapping new dynamic system the axis $z = 0$ will be the invariant axis (that mean it will consist of trajectories of this system). On this axis now will lie the singular points $O_i\ (u_i, 0),\ i = \overline{0, m}$, where $u_i,\ i = \overline{1, m}$ are all existing real roots of the polynomial $P(u)$, and $u_0 = 0$; the same time may exist $i_0 \in \{1, \ldots, m\}$: $u_{i_0} = 0$. Further we are going to name such singular points the infinitely remote (IR) points of the 1st kind for the dynamic systems described by the equations (1).

The second Poincare transformation

$$x = \frac{v}{z}, \quad y = \frac{1}{z} \quad \left(v = \frac{x}{y}, \quad z = \frac{1}{y}\right)$$

somehow similarly maps a phase real plane $R^2_{x,y}$ anambiguosly onto a sphere $\sum$, where the diametrically opposite points are also considered to be identified. At first we again consider the Poincare sphere $\sum$ without it`s equator. The second Poincare transformation translates each Eq.(1) - system into a new dynamic system. This newly obtained dynamic system in the new coordinates of the second Poincare mapping $\tau, v, z$ looks like:

$$\frac{dv}{d\tau} = -X(v,1) + Y(v,1)vz, \ \frac{dz}{d\tau} = Y(v,1)z^2.$$

It is considered and determined on the whole Poincare sphere $\sum$, as well as on the whole $(v,z)-$ plane $\hat{\alpha}$, which is tangent to a sphere $\sum$ at a point $D = (0,1,0)$ [1].

A set described with the equation $z = 0$ is the invariant set for the new dynamic system. On this set will lie the singular points $(v_0, 0)$, where $v_0$ is any real root of the polynomial $X(v,1) \equiv p_3 + p_2 v + p_1 v^2 + p_0 v^3$. We could name those singular points the infinitely remote points (IR-points) of the 2nd kind for the dynamic systems described by the equations (1), but every singular point of the 2nd kind, for which $v_0 \neq 0$, naturally coincide to someone among the IR-points of the 1st kind, and precisely to the point $\left(\frac{1}{v_0}, 0\right)$,

while $v_0 = 0$ won't be a root for the polynomial $X(x, 1)$, since $X(0, 1) = p_3 \neq 0$ for the dynamic system described by the equations (1). As a result, we can formulate the

**Corollary C.** Any dynamic system belonging to the broad family described by the equations (1) has infinitely remote singular points only of the 1st kind.

Using the orthogonal projection, further we proceed a mapping of a closed lower semi sphere $\bar{H}$ of a Poincare sphere $\sum$ onto a real plane *x, y*. As a result of this mapping the open part *H* of a lower semi sphere of a Poincare sphere $\sum$ maps one-to-one onto the open Poincare disk $\Omega$, and an equator of the Poincare sphere $\sum$ (its boundary) *E* maps onto a very circle, a bondary of the Poincare disk, $\Gamma=\partial\Omega$. =>

1) All the own trajectories of any dynamic system described by the equations (1), including the isolated finite singular point *O (0, 0)*, now displayed into the disk **Ω**, and they are filling the Poincare disk. They are called trajectories of the system under consideration (the Eq.(1) – system) in the Poincare disk **Ω**.

2) The infinitely remote trajectories, including the infinitely remote singular points, are displayed now on a boundary *Γ* of a Poincare disk **Ω**, and they are filling this boundary. They are called trajectories of the system under consideration (the Eq.(1) – system) on the Poincare disk's boundary – the circle *Γ*.

Each infinitely remote singular point $O_i\ (u_i, 0)$ of the Eq.(1) – system, $i \in \{1, \dots, m\}$, now corresponds to the two diametrically opposite points situated on the Poincare disk's boundary – the circle *Γ* .

$O_i^{\pm}(u_i, 0)$: $O_i^+\ (O_i^-) \in \Gamma^{+(-)} := \Gamma|_{x>0\ (x<0)}$.

$\forall\, i \in \{1, \dots, m\}$ for the point $O_i^+\ (O_i^-)$ we now need to introduce the necessary and important notations.

1) $O_i^+\ (O_i^-)$—curve be a semi trajectory of the Eq.(1) – system in the Poincare disk **Ω**, which is adjacent to a singular point $O_i^{+(-)}$. It starts in the some ordinary point $p \in \Omega$.

2) Notations of nodal and saddle bandles *N*, *S*, which are adjacent to the singular point $O_i^+\ (O_i^-)$ from the Poincare disk **Ω**, will correspond to the notations introduced for the isolated singular point *O (0, 0).*

3) We introduce the notations of the words $A_i^+\ (A_i^-)$ containing letters *N*, *S*, which are fixing the order of bandles of $O_i^+\ (O_i^-)$-curves at a semi circumvention of the point $O_i^+\ (O_i^-)$ in the Poincare disk **Ω**. This semi circumvention is proceeded in the direction of increasing of *u*.

We are going to describe the topological type of the singular point $O_i^+\ (O_i^-)$ with a word $A_i^+\ (A_i^-)$, while we describe the topological type of a singular point $O_i$ with words $A_i^{\pm}$.

For the topological types of the infinitely remote singular points $O_0^{\pm}\ (0,0)$ of the family of dynamic systems described by the equations (1) the following theorem is formulated and proved.

**Theorem B.** If we consider a number $u = 0$ as the root of a polynomial $P(u)$ of the Eq.(1) – system, having the multiplicity $k \in \{0, \dots, 3\}$, then the words $A_0^{\pm}$ , used to describe the topologycal types of the infinitely remote singular points $O_0^{\pm}\ (0,0)$ of the system, may only have the forms, written in the Table 3 below. The precise form will depend on the sign of a number $ap_k$ and the value of *k* (here *a* and $p_k$ are the coefficients of the system) [5]**.**

Table 3. Topological types of infinitely remote singular points $O_0^{\pm}$ (0,0).

| $k$ | $ap_k$ | $A_0^+$ | $A_0^-$ |
|---|---|---|---|
| 0 | 0 | $N$ | $N$ |
| 0, 2 | + (-) | $N_+(N_-)$ | $N_-(N_+)$ |
| 1, 3 | + (-) | $N_-N_+(\emptyset)$ | $\emptyset(N_-N_+)$ |

**Corollary D**. Infinitely remote singular points $O_0^{\pm}$ of any dynamic system described by the equations (1) has no separatrices.

**Theorem C**. If we consider a real number $u_i$ $(\neq 0)$ as the root of a polynomial $P(u)$ of the Eq.(1) – system, having the multiplicity $k_i \in \{1,2,3\}$, then the words $A_i^{\pm}$, as well as the value $g_i = P^{(ki)}(u_i)Q(u_i) \neq 0$, which describe and determine the topologycal types of the infinitely remote singular points $O_i^{\pm}$ $(u_i, 0)$ of the system under consideration, may have only forms written in the Table 4 below. The precise form will depend on the value of $k_i$ and signs of numbers $u_i$ and $g_i$, [5].

Table 4. Topological types of the infinitely remote points $O_i^{\pm}$ $(u_i, 0)$, $i \in \{1, \dots, m\}$.

| $u_i$ | $k_i$ | $g_i$ | $A_i^+$ | $A_i^-$ |
|---|---|---|---|---|
| +(-) | 1, 3 | + | $N_+(N_-)$ | $S_-(S_+)$ |
| +(-) | 1, 3 | _ | $S_-(S_+)$ | $N_+(N_-)$ |
| +(-) | 2 | + | $S_-N_+(\emptyset)$ | $\emptyset(N_-S_+)$ |
| +(-) | 2 | _ | $\emptyset(N_-S_+)$ | $S_-N_+(\emptyset)$ |

**Corollary E**. We can conclude using the Theorems B and C, that for the infinitely remote singular points of the family of dynamic systems described by the equations (1), only the finite number, and namely 13 (thirteen) separate topological types are existing. The thorough detailed study of those 13 topological types makes clear, that the infinitely remote singular points of every Eq.(1) - system may have only $m$ separatrices: one separatrice for every singular point $O_i$ $(u_i, 0), i = \overline{1, m}$.

**Note C**. In the Tables 3, 4 the lower sign index «+» or «-» of every nodal or saddle bundle $N$ or $S$, shows will the given bundle adjust to the singular point $O_i^+$ (or to the point $O_i^-$) from the side where $u > u_i$ or from the side where $u < u_i$ of the isocline described with the equation $u = u_i$.

In the Table 3, see line 1, a nodal bundle $N$ has no lower sign index at all, that means that it contains $O_i^+$-curves ($O_i^-$-curves) in each domain where $|u| > 0$ [5-9]].

## Conclusions

The present paper is written as a result of the original study in the field of the qualitative theory of differential equations and dynamic systems.

The main aim of the study is to investigate and construct all different in the topological and topo-dynamical sense phase portraits of a broad family of dynamic systems. The systems belonging to this family are clearly characterized with their reciprocal polynomial right parts. A total broad family has numerical subfamilies, interesting in the different applications.

The whole family of dynamic systems under consideration was studied using the first and the second Poincare mappings on the Poincare sphere, in the Poincare disk and on its boundary - the Poincare circle. We have constructed all topologically different phase portraits which are possible for the systems of this family, about 250 types of phase portraits. All those types were constructed basing on the theoretical precise proofs. [6, 7].

For this aim it was necessary to investigate all finite and infinitely remote singular points of systems under consideration. Purposes of such an investigation demanded developing of new powerful special and totally new research methods. [8-10].

## Recommendations

Despite of the theoretical nature of the present investigation, due to obtained pioneer results and developed in this study new research methods [11-15], authors consider the work to be useful for different applied studies of dynamic systems having polynomial right parts [16, 17].. The paper will be interesting and addressed to students, postgraduates and scientific researchers.

## References

Andronov, A.A., Leontovich, E.A., Gordon, I.I., & Maier, A.G. (1973). *Qualitative theory of second-order dynamic systems*. New York, NY: Wiley.

Andreev, A.F., & Andreeva, I.A. (1997). On limit and separatrix cycles of a certain quasi quadratic system. *Differential Equations, 33 (*5*),* 702 – 703.

Andreev, A.F., & Andreeva, I.A. (2007). Local study of a family of planar cubic systems. *Vestnik St. Petersburg University: Ser.1. Mathematics, Mechanics, Astronomy, 2*, 11- 16. DOI: 10.3103/S1063454107020021, EID: 2-s2.0-84859730890.

Andreev, A.F., Andreeva, I.A., Detchenya, L.V., Makovetskaya, T.V., & Sadovskii, A.P. (2017). Nilpotent Centers of Cubic Systems. *Differential Equations*, *53*(8), 1003 - 1008. DOI: 10.1134/S0012266117080018, EID: 2-s2.0-85029534241.

Andreev, A.F., & Andreeva, I.A. (2007). Phase flows of one family of cubic systems in a Poincare circle. I. *Differential Equations and Control*, *4*, 17-26.

Andreev, A.F., & Andreeva, I.A. (2008). Phase flows of one family of cubic systems in a Poincare circle. II. *Differential Equations and Control*, *1,* 1 - 13.

Andreev, A.F., & Andreeva, I.A. (2008). Phase flows of one family of cubic systems in a Poincare circle. III. *Differential Equations and Control/*, *3*, 39 - 54.

Andreev, A.F., & Andreeva, I.A. (2009). Phase flows of one family of cubic systems in a Poincare circle. $IV_1$. *Differential Equations and Control, 4,* 181 - 213.

Andreev, A.F., &Andreeva, I.A. (2010). Phase flows of one family of cubic systems in a Poincare circle. $IV_2$. *Differential Equations and Control, 4*, 6- 17.

Andreev, A.F., & Andreeva, I.A. (2017). Investigation of a Family of Cubic Dynamic Systems. *Vibroengineering Procedia*, *15,* 88 – 93. DOI: 10.21595/vp.2017.19389.

Andreev, A.F., & Andreeva, I.A. (2017). Phase Portraits of One Family of Cubic Systems in a Poincare Circle.I. *Vestnik RAEN*, 17, 4, 8 – 18. ISSN 1682-1696.

Andreev, A.F., &Andreeva, I.A. (2018) On a Behavior of Trajectories of a Certain Family of Cubic Dynamic Systems in a Poincare Circle, *IOP Journal of Physics, Conference Series,* 1141.

Andreeva, I.A. (2018) Phase Portraits of Cubic Dynamic Systems in a Poincare Circle. *(A Chapter in the Monograph "Differential Equations- Theory and Current Research". IntechOpen, UK,* ISBN 978-953-51-6120-2.

Andreev, A.F., & Andreeva, I.A. (2018). Phase Portraits of a Family of Cubic Dynamic Systems in a Poincare Circle.II. *Vestnik RAEN*, 18, 4, 11 – 15. ISSN 1682-1696.

Andreeva, I.A., &Efimova, T.O. (2019) Phase Portraits of a Special Class of Dynamic Systems in a Poincare Circle, *IOP Journal of Physics, Conference Series*, 1236.

Aksenova, O.A., & Khalidov, I.A. (2016). Analytic Model of the Effect of Poly-Gaussian Roughness on Rarefied Gas Flow near the Surface, *Rarefied Gas Dynamics*, *AIP Conference Proceedings, 1786, American Institute of Physics, Melville, NY,* 1000071 - 1000078.

Aksenova, O.A., & Khalidov, I.A. (2016). Unstable Rarefied Gas Flow Conditions in a Channel, *Rarefied Gas Dynamics, AIP Conference Proceedings 1786, American Institute of Physics, Melville, NY*, 1000091-1000097.

## Authors Information

**Irina Andreeva**
Peter the Great St. Petersburg Polytechnic University
195251 St. Petersburg, Polytechnicheskaya, 29.
Russian Federation
Contact E-mail: *irandr2@gmail.com*

**Tatiana Efimova**
Peter the Great St. Petersburg Polytechnic University
195251 St. Petersburg, Polytechnicheskaya, 29.
Russian Federation