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Solutions and Solution Behavior of Difference Equation Systems

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Abstract: The behaviour of the solutions of the following system of difference equations is examined.
$$x_{n+1} = \max\left\{\frac{1}{x_{n-2}}, \frac{y_{n-2}}{x_{n-2}}\right\},\ y_{n+1} = \max\left\{\frac{1}{y_{n-2}}, \frac{x_{n-2}}{y_{n-2}}\right\} \qquad (1)$$
where the initial conditions are positive real numbers. In this study, the solutions of the initial conditions in the range (0,1) and the behavior of the solutions were examined. Investigations on the recent studies show that researches nature of nonlinear difference equations have been an object of great interest. Although difference equations are relatively simple in form, it is, unfortunately, extremely difficult to understand thoroughly the periodic and non periodic behavior of their solutions. Nonlinear difference equations of the max type has been investigated by many authors. These equations are particularly utilized in various fields, including population dynamics, economics, finance, and control systems. Max-type difference equations are not only interesting from a theoretical perspective but also possess a wide range of practical applications. They are frequently employed to model discrete dynamical systems in control theory, economic fluctuations, and biological population models with delayed feedback mechanisms. Understanding the periodic behavior and stability of such systems provides vital insights into the long-term dynamics of these real-world phenomena. In addition, Mathematica software was employed to generate the graphical illustrations and numerical examples presented in this study.

Keywords: Difference equation, Maximum operations, Semicycle

Introduction

Recently, there has been a great interest in studying the periodic nature of nonlinear difference equations. Although difference equations are relatively simple in form, it is, unfortunately, extremely diffucult to understand thorougly the periodic behavior of their solutions. The periodic nature of nonlinear difference equations of the max type has been investigated by many authors. See for example [1-28].

Amleh (1998), in her doctoral dissertation conducted under the supervision of G. Ladas, addressed three distinct topics in difference equations. In the first chapter, he demonstrated that the solutions to the difference equation

$$x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{B}{x_{n-1}}\right\}$$

are periodic for non-zero real parameters and initial conditions.

Janowski et al. (1998), in their study;

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$$x_{n+1} = \frac{\max\left\{x_n^k, A\right\}}{x_{n-1}}$$

they investigated the boundedness and oscillation properties of solutions to the rational difference equation with maximum.

In his doctoral dissertation, Valicenti (1999) studied the periodicity and global asymptotic stability of solutions to

the Lyness difference equation $x_{n+1} = \frac{\max\left\{a_n x_n, b_n\right\}}{x_{n-1}}$ and the difference equation with maximum

$$x_{n+1} = \frac{a_n x_n + b_n}{x_{n-1}}.$$

In his doctoral dissertation, Teixeira (2000) first investigated the periodicity of solutions to the difference equation

$x_{n+1} = \frac{\max\left\{x_n, A\right\}}{x_n x_{n-1}}$ where A is an arbitrary real number and the initial conditions are non-zero real numbers.

Papaschinopoulos and Hatzifilippidis (2001) examined the continuity, boundedness, and periodicity of positive solutions for the difference equation

$x_{n+1} = \frac{\max\left\{a_n\left(\prod_{i=n-k+1}^{n} x_i\right), b_n\right\}}{\prod_{i=n-k}^{n} x_i}$ assuming positive sequences for the coefficients and positive values for the initial conditions.

In their study on the periodicity of the difference equation $x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{B}{x_{n-2}}\right\}$, Mishev et al. (2002) proved that all positive solutions of the equation are eventually periodic, assuming that the parameters and initial conditions are positive real values.

In the first of his two studies, Voulov (2002) solved an open problem posed by G. Ladas. In this work, he demonstrated that all solutions of the difference equation $x_n = \max\left\{\frac{A}{x_{n-1}}, \frac{B}{x_{n-3}}, \frac{C}{x_{n-5}}\right\}$ are periodic for $A + B + C > 0$, assuming the parameters are non-negative real numbers.

Feuer (2003) examined the periodic nature of solutions to the maximum-type Lyness difference equation $x_{n+1} = \frac{\max\left\{x_n^k, A\right\}}{x_n^\ell x_{n-1}}$, where A is a positive real number and both the parameters and initial conditions are taken as arbitrary real numbers.

In their 2003 study, Papaschinopoulos et al. (2003) investigated the solutions, their periodicity, and the invariant intervals of the difference equation $x_{n+1} = \frac{\max\left\{x_n, A\right\}}{x_n x_{n-1}}$, which had previously been studied by Feuer.

In their study, Patula and Voulov (2004) investigated the periodicity of solutions to the difference equation $x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-2}}\right\}$, where A_n and B_n are positive sequences with period 3.

In their 2005 study, Çinar et al.(2005) investigated the periodicity of positive solutions to the difference equation $x_{n+1} = \min\left\{\frac{A}{x_n}, \frac{B}{x_{n-2}}\right\}$,for $A, B > 0$ and non-zero initial conditions. Furthermore, they generalized this equation to $x_{n+1} = \min\left\{\frac{A}{x_n x_{n-1} \ldots x_{n-k}}, \frac{B}{x_{n(k+2)} \ldots x_{n-(2k+2)}}\right\}$ and examined the periodicity of its positive solutions as well.

In their 2006 study, Simsek et al. (2006) investigated the periodicity of solutions to the difference equation $x_{n+1} = \max\left\{\frac{1}{x_{n-1}}, x_{n-1}\right\}$ under positive initial conditions.

In their 2009 study, Simsek et al. (2009) investigated the solutions of the system of difference equations with maximum

$x_{n+1} = max\left\{\frac{1}{x_n}, \frac{y_n}{x_n}\right\}$ and $y_{n+1} = max\left\{\frac{1}{y_n}, \frac{x_n}{y_n}\right\}$ by selecting positive initial conditions.

In their 2009 study, Simsek et al. (2009) investigated the solutions of the system of difference equations with maximum

$x_{n+1} = max\left\{\frac{A}{x_n}, \frac{y_n}{x_n}\right\}$ and $y_{n+1} = max\left\{\frac{A}{y_n}, \frac{x_n}{y_n}\right\}$ assuming that both the parameter *A* and the initial conditions are positive.

In their 2010 study, Simsek et al. (2010) investigated the behaviors of solutions for the system of difference equations with maximum $x_{n+1} = max\left\{\frac{1}{x_{n-1}}, \frac{y_{n-1}}{x_{n-1}}\right\}$ and $y_{n+1} = max\left\{\frac{1}{y_{n-1}}, \frac{x_{n-1}}{y_{n-1}}\right\}$ assuming positive initial conditions.

In their 2006 study, Yan et al. demonstrated that the solutions of the difference equation $x_n = \max\left\{\frac{1}{x_{n-1}^{\alpha}}, \frac{A}{x_{n-2}}\right\}$ are periodic with 4 for the initial conditions $0 < \alpha < 1, A > 0, A \le 1,\ A > 1$ and $x_{-2}, x_{-1}, x_0 \in (0,\infty)$.

Definition 1: Let I be an interval of real numbers and let $f : I^{s+1} \to I$ be a continuously differentiable function where s is a non-negative integer. Consider the difference equation

$$x_{n+1} = f\left(x_n, x_{n-1}, \ldots, x_{n-s}\right) \text{ for n } = 0,1,2, \tag{1}$$

with the initial values $x_{-s}, \ldots, x_0 \in I$. A point $\bar{x}$ called an equilibrium point of Eq.(1) if

$\bar{x} = f(\bar{x}, \ldots, \bar{x})$

Definition 2: A positive semicycle of a solution $\left\{x_n\right\}_{n=-s}^{\infty}$ Eq.(1) consists of a string of terms $\left\{x_l, x_{l+1}, \ldots, x_m\right\}$ all greater than or equal to equilibrium $\bar{x}$ with $l \geq -s$ and $m \leq \infty$ such that either $l = -s$ or $l > -s$ and $x_{l-1} < \bar{x}$ and either $m = \infty$ or $m \leq \infty$ and $x_{m+1} < \bar{x}$

Definition 3: A negative semicycle of a solution $\{x_n\}_{n=-s}^{\infty}$ of Eq.(1) consists of a string of terms $\{x_l, x_{l+1}, \ldots, x_m\}$ all less than or equal to equilibrium $\bar{x}$ with $l \geq -s$ and $m \leq \infty$ such that either $l=-s$ or $l>-s$ and $x_{l-1} < \bar{x}$ and either $m=\infty$ or $m \leq \infty$ and $x_{m+1} < \bar{x}$

Definition 4: The Fibonacci numbers are defined by the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$, with initial conditions $f_1 = 1$ *and* $f_2 = 1$.

Main Results

To this end, the statements and proofs of lemmas and theorems are presented for various cases that may be encountered in equation (1).

Let $\bar{x}$ and $\bar{y}$ be the unique positive equilibrium of the (1), then clearly

$$\bar{x} = \max\left\{\frac{1}{\bar{x}}, \frac{\bar{y}}{\bar{x}}\right\} \text{ and } \bar{y} = \max\left\{\frac{1}{\bar{y}}, \frac{\bar{x}}{\bar{y}}\right\}$$

$$\bar{x} = \frac{1}{\bar{x}} \text{ or } \bar{x} = \frac{\bar{y}}{\bar{x}} \text{ and } \bar{y} = \frac{1}{\bar{y}} \text{ or } \bar{y} = \frac{\bar{x}}{\bar{y}}$$

and

$$(\bar{x})^2 = 1 \text{ and } (\bar{y})^2 = 1$$

We can obtain $\bar{x} = 1$ and $\bar{y} = 1$.

***Lemma 1*:** For equation (1),

$$0 < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-1} < y_0 < 1$$

the following statements are true for the solutions x_n and y_n with respect to the initial conditions:

a) Every positive semicycle consists of three terms.
b) Every negative semicycle consists of three terms.
c) Solution x_n, every positive semicycle of length three is followed by a negative semicycle of length three. Solution y_n, for $n \geq 4$; every positive semicycle of length three is followed by a negative semicycle of length three.
d) Solution x_n, for $n \geq 4$; every negative semicycle of length three is followed by a positive semicycle of length three. Solution y_n, for $n \geq 7$; every negative semicycle of length three is followed by a positive semicycle of length three.

Proof: Let us present the proof for Case 1.

$$0 < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-1} < y_0 < 1$$

Depending on the initial conditions:

$$x_1 = \max\left\{\frac{1}{x_{-2}}, \frac{y_{-2}}{x_{-2}}\right\} = \frac{1}{x_{-2}} > \bar{x}$$

$$y_1 = \max\left\{\frac{1}{y_{-2}}, \frac{x_{-2}}{y_{-2}}\right\} = \frac{1}{y_{-2}} > \bar{y}$$

$$x_2 = \max\left\{\frac{1}{x_{-1}}, \frac{y_{-1}}{x_{-1}}\right\} = \frac{1}{x_{-1}} > \bar{x}$$

$$y_2 = \max\left\{\frac{1}{y_{-1}}, \frac{x_{-1}}{y_{-1}}\right\} = \frac{1}{y_{-1}} > \bar{y}$$

$$x_3 = \max\left\{\frac{1}{x_0}, \frac{y_0}{x_0}\right\} = \frac{1}{x_0} > \bar{x}$$

$$y_3 = \max\left\{\frac{1}{y_0}, \frac{x_0}{y_0}\right\} = \frac{1}{y_0} > \bar{y}$$

$$x_4 = \max\left\{\frac{1}{x_1}, \frac{y_1}{x_1}\right\} = \left\{x_{-2}, \frac{x_{-2}}{y_{-2}}\right\} = \frac{x_{-2}}{y_{-2}} < \bar{x}$$

$$y_4 = \max\left\{\frac{1}{y_1}, \frac{x_1}{y_1}\right\} = \left\{y_{-2}, \frac{y_{-2}}{x_{-2}}\right\} = \frac{y_{-2}}{x_{-2}} > \bar{y}$$

$$x_5 = \max\left\{\frac{1}{x_2}, \frac{y_2}{x_2}\right\} = \left\{x_{-1}, \frac{x_{-1}}{y_{-1}}\right\} = \frac{x_{-1}}{y_{-1}} < \bar{x}$$

$$y_5 = \max\left\{\frac{1}{y_2}, \frac{x_2}{y_2}\right\} = \left\{y_{-1}, \frac{y_{-1}}{x_{-1}}\right\} = \frac{y_{-1}}{x_{-1}} > \bar{y}$$

$$x_6 = \max\left\{\frac{1}{x_3}, \frac{y_3}{x_3}\right\} = \left\{x_0, \frac{x_0^2}{y_0^2}\right\} = \frac{x_0}{y_0} < \bar{x}$$

$$y_6 = \max\left\{\frac{1}{y_3}, \frac{x_3}{y_3}\right\} = \left\{y_0, \frac{y_0}{x_0}\right\} = \frac{y_0}{x_0} > \bar{y}$$

$$x_7 = \max\left\{\frac{1}{x_4}, \frac{y_4}{x_4}\right\} = \left\{\frac{y_{-2}}{x_{-2}}, \frac{y_{-2}^2}{x_{-2}^2}\right\} = \frac{y_{-2}^2}{x_{-2}^2} > \bar{x}$$

$$y_7 = \max\left\{\frac{1}{y_4}, \frac{x_4}{y_4}\right\} = \left\{\frac{x_{-2}}{y_{-2}}, \frac{x_{-2}^2}{y_{-2}^2}\right\} = \frac{x_{-2}}{y_{-2}} < \bar{y}$$

$$x_8 = \max\left\{\frac{1}{x_5}, \frac{y_5}{x_5}\right\} = \left\{\frac{y_{-1}}{x_{-1}}, \frac{y_{-1}^2}{x_{-1}^2}\right\} = \frac{y_{-1}^2}{x_{-1}^2} > \bar{x}$$

$$y_8 = \max\left\{\frac{1}{y_5}, \frac{x_5}{y_5}\right\} = \left\{\frac{x_{-1}}{y_{-1}}, \frac{x_{-1}^2}{y_{-1}^2}\right\} = \frac{x_{-1}}{y_{-1}} < \overline{y}$$

$$x_9 = \max\left\{\frac{1}{x_6}, \frac{y_6}{x_6}\right\} = \left\{\frac{y_0}{x_0}, \frac{y_0^2}{x_0^2}\right\} = \frac{y_0^2}{x_0^2} > \overline{x}$$

$$y_9 = \max\left\{\frac{1}{y_6}, \frac{x_6}{y_6}\right\} = \left\{\frac{x_0}{y_0}, \frac{x_0^2}{y_0^2}\right\} = \frac{x_0}{y_0} < \overline{y}$$

$$x_{10} = \max\left\{\frac{1}{x_7}, \frac{y_7}{x_7}\right\} = \left\{\frac{x_{-2}^2}{y_{-2}^2}, \frac{x_{-2}^3}{y_{-2}^3}\right\} = \frac{x_{-2}^2}{y_{-2}^2} < \overline{x}$$

$$y_{10} = \max\left\{\frac{1}{y_7}, \frac{x_7}{y_7}\right\} = \left\{\frac{y_{-2}}{x_{-2}}, \frac{y_{-2}^3}{x_{-2}^3}\right\} = \frac{y_{-2}^3}{x_{-2}^3} > \overline{y}$$

$$x_{11} = \max\left\{\frac{1}{x_8}, \frac{y_8}{x_8}\right\} = \left\{\frac{x_{-1}^2}{y_{-1}^2}, \frac{x_{-1}^3}{y_{-1}^3}\right\} = \frac{x_{-1}^2}{y_{-1}^2} < \overline{x}$$

$$y_{11} = \max\left\{\frac{1}{y_8}, \frac{x_8}{y_8}\right\} = \left\{\frac{y_{-1}}{x_{-1}}, \frac{y_{-1}^3}{x_{-1}^3}\right\} = \frac{y_{-1}^3}{x_{-1}^3} > \overline{y}$$

$$x_{12} = \max\left\{\frac{1}{x_9}, \frac{y_9}{x_9}\right\} = \left\{\frac{x_0^2}{y_0^2}, \frac{x_0^3}{y_0^3}\right\} = \frac{x_0^2}{y_0^2} < \overline{x}$$

$$y_{12} = \max\left\{\frac{1}{y_9}, \frac{x_9}{y_9}\right\} = \left\{\frac{y_0}{x_0}, \frac{y_0^3}{x_0^3}\right\} = \frac{y_0^3}{x_0^3} > \overline{y}$$

$$\vdots$$

$x_1 > \overline{x}$, $x_2 > \overline{x}$, $x_3 > \overline{x}$, $x_4 < \overline{x}$, $x_5 < \overline{x}$, $x_6 < \overline{x}$, $x_7 > \overline{x}$, $x_8 > \overline{x}$, $x_9 > \overline{x}$,

$x_{10} < \overline{x}$, $x_{11} < \overline{x}$, $x_{12} < \overline{x}$, …

Thus, as can be seen from this, the solutions x_n follow the pattern PPPNNNPPPNNN...

It is observed that every positive semicycle consists of three terms.

It is observed that every negative semicycle consists of three terms.

It is seen from the solutions x_n that every positive semicycle of length three is followed by a negative semicycle of length three.

For $n \geq 4$, every negative semicycle of length three is followed by a positive semicycle of length three.

$y_1 > \bar{y}, y_2 > \bar{y}, y_3 > \bar{y}, y_4 > \bar{y}, y_5 > \bar{y}, y_6 > \bar{y}, y_7 < \bar{y}, y_8 < \bar{y}, y_9 < \bar{y}, y_{10} > \bar{y},\ y_{11} > \bar{y},$
$y_{12} > \bar{y},$

Thus, as can be seen from this, the solutions y_n follow the pattern PPPPPPNNNPPPNNN...

It is observed that every positive semicycle consists of three terms.

It is observed that every negative semicycle consists of three terms.

For $n \geq 4$, every positive semicycle of length three is followed by a negative semicycle of length three.

For $n \geq 7$, it is seen from the solutions y_n that every negative semicycle of length three is followed by a positive semicycle of length three.

This completes the proof of Lemma 1.

According to the following initial conditions

$0 < y_{-2} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_0 < 1$, $0 < x_{-2} < x_{-1} < y_0 < x_0 < y_{-2} < y_{-1} < 1$,

$0 < y_{-2} < x_{-2} < x_{-1} < y_{-1} < y_0 < x_0 < 1$, $0 < x_{-2} < y_{-2} < y_{-1} < x_{-1} < x_0 < y_0 < 1$,

$0 < y_{-2} < x_{-2} < y_{-1} < x_{-1} < x_0 < y_0 < 1$, $0 < x_{-2} < y_{-2} < y_{-1} < x_{-1} < y_0 < x_0 < 1$,

$0 < y_{-2} < x_{-2} < y_{-1} < x_{-1} < y_0 < x_0 < 1$.

The proof for the solutions x_n and y_n is carried out similarly to the proof of Lemma 1.

Theorem 1: $0 < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-1} < y_0 < 1$

Where

$x_1 = \dfrac{1}{x_{-2}},\ x_2 = \dfrac{1}{x_{-1}},\ x_3 = \dfrac{1}{x_0}$ and $y_1 = \dfrac{1}{y_{-2}},\ y_2 = \dfrac{1}{y_{-1}},\ y_3 = \dfrac{1}{y_0}$

the solutions of equation (1):

a)
$$x_{6n+4} = \left(\frac{x_{-2}}{y_{-2}}\right)^{f(2n+1)},\ x_{6n+5} = \left(\frac{x_{-1}}{y_{-1}}\right)^{f(2n+1)},\ x_{6n+6} = \left(\frac{x_0}{y_0}\right)^{f(2n+1)},$$
$$x_{6n+7} = \left(\frac{y_{-2}}{x_{-2}}\right)^{f(2n+3)},\ x_{6n+8} = \left(\frac{y_{-1}}{x_{-1}}\right)^{f(2n+3)},\ x_{6n+9} = \left(\frac{y_0}{x_0}\right)^{f(2n+3)}$$

b)
$$y_{6n+4} = \left(\frac{y_{-2}}{x_{-2}}\right)^{f(2n+2)},\ y_{6n+5} = \left(\frac{y_{-1}}{x_{-1}}\right)^{f(2n+2)},\ y_{6n+6} = \left(\frac{y_0}{x_0}\right)^{f(2n+2)},$$
$$y_{6n+7} = \left(\frac{x_{-2}}{y_{-2}}\right)^{f(2n+2)},\ y_{6n+8} = \left(\frac{x_{-1}}{y_{-1}}\right)^{f(2n+2)},\ y_{6n+9} = \left(\frac{x_0}{y_0}\right)^{f(2n+2)}$$

is of the form. For n = 0,1,2, …, the above equations are true.

Proof: Let us show the validity of the above statements by mathematical induction.

For $n = 0$,

$$x_1 = \max\left\{\frac{1}{x_{-2}}, \frac{y_{-2}}{x_{-2}}\right\} = \frac{1}{x_{-2}} > \bar{x}$$

$$y_1 = \max\left\{\frac{1}{y_{-2}}, \frac{x_{-2}}{y_{-2}}\right\} = \frac{1}{y_{-2}} > \bar{y}$$

$$x_2 = \max\left\{\frac{1}{x_{-1}}, \frac{y_{-1}}{x_{-1}}\right\} = \frac{1}{x_{-1}} > \bar{x}$$

$$y_2 = \max\left\{\frac{1}{y_{-1}}, \frac{x_{-1}}{y_{-1}}\right\} = \frac{1}{y_{-1}} > \bar{y}$$

$$x_3 = \max\left\{\frac{1}{x_0}, \frac{y_0}{x_0}\right\} = \frac{1}{x_0} > \bar{x}$$

$$y_3 = \max\left\{\frac{1}{y_0}, \frac{x_0}{y_0}\right\} = \frac{1}{y_0} > \bar{y}$$

$$x_4 = \max\left\{\frac{1}{x_1}, \frac{y_1}{x_1}\right\} = \left\{x_{-2}, \frac{x_{-2}}{y_{-2}}\right\} = \frac{x_{-2}}{y_{-2}} < \bar{x}$$

$$y_4 = \max\left\{\frac{1}{y_1}, \frac{x_1}{y_1}\right\} = \left\{y_{-2}, \frac{y_{-2}}{x_{-2}}\right\} = \frac{y_{-2}}{x_{-2}} > \bar{y}$$

$$x_5 = \max\left\{\frac{1}{x_2}, \frac{y_2}{x_2}\right\} = \left\{x_{-1}, \frac{x_{-1}}{y_{-1}}\right\} = \frac{x_{-1}}{y_{-1}} < \bar{x}$$

$$y_5 = \max\left\{\frac{1}{y_2}, \frac{x_2}{y_2}\right\} = \left\{y_{-1}, \frac{y_{-1}}{x_{-1}}\right\} = \frac{y_{-1}}{x_{-1}} > \bar{y}$$

$$x_6 = \max\left\{\frac{1}{x_3}, \frac{y_3}{x_3}\right\} = \left\{x_0, \frac{x_0^2}{y_0^2}\right\} = \frac{x_0}{y_0} < \bar{x}$$

$$y_6 = \max\left\{\frac{1}{y_3}, \frac{x_3}{y_3}\right\} = \left\{y_0, \frac{y_0}{x_0}\right\} = \frac{y_0}{x_0} > \bar{y}$$

$$x_7 = \max\left\{\frac{1}{x_4}, \frac{y_4}{x_4}\right\} = \left\{\frac{y_{-2}}{x_{-2}}, \frac{y_{-2}^2}{x_{-2}^2}\right\} = \frac{y_{-2}^2}{x_{-2}^2} > \overline{x}$$

$$y_7 = \max\left\{\frac{1}{y_4}, \frac{x_4}{y_4}\right\} = \left\{\frac{x_{-2}}{y_{-2}}, \frac{x_{-2}^2}{y_{-2}^2}\right\} = \frac{x_{-2}}{y_{-2}} < \overline{y}$$

$$x_8 = \max\left\{\frac{1}{x_5}, \frac{y_5}{x_5}\right\} = \left\{\frac{y_{-1}}{x_{-1}}, \frac{y_{-1}^2}{x_{-1}^2}\right\} = \frac{y_{-1}^2}{x_{-1}^2} > \overline{x}$$

$$y_8 = \max\left\{\frac{1}{y_5}, \frac{x_5}{y_5}\right\} = \left\{\frac{x_{-1}}{y_{-1}}, \frac{x_{-1}^2}{y_{-1}^2}\right\} = \frac{x_{-1}}{y_{-1}} < \overline{y}$$

$$x_9 = \max\left\{\frac{1}{x_6}, \frac{y_6}{x_6}\right\} = \left\{\frac{y_0}{x_0}, \frac{y_0^2}{x_0^2}\right\} = \frac{y_0^2}{x_0^2} > \overline{x}$$

$$y_9 = \max\left\{\frac{1}{y_6}, \frac{x_6}{y_6}\right\} = \left\{\frac{x_0}{y_0}, \frac{x_0^2}{y_0^2}\right\} = \frac{x_0}{y_0} < \overline{y}$$

$$x_{10} = \max\left\{\frac{1}{x_7}, \frac{y_7}{x_7}\right\} = \left\{\frac{x_{-2}^2}{y_{-2}^2}, \frac{x_{-2}^3}{y_{-2}^3}\right\} = \frac{x_{-2}^2}{y_{-2}^2} < \overline{x}$$

$$y_{10} = \max\left\{\frac{1}{y_7}, \frac{x_7}{y_7}\right\} = \left\{\frac{y_{-2}}{x_{-2}}, \frac{y_{-2}^3}{x_{-2}^3}\right\} = \frac{y_{-2}^3}{x_{-2}^3} > \overline{y}$$

$$x_{11} = \max\left\{\frac{1}{x_8}, \frac{y_8}{x_8}\right\} = \left\{\frac{x_{-1}^2}{y_{-1}^2}, \frac{x_{-1}^3}{y_{-1}^3}\right\} = \frac{x_{-1}^2}{y_{-1}^2} < \overline{x}$$

$$y_{11} = \max\left\{\frac{1}{y_8}, \frac{x_8}{y_8}\right\} = \left\{\frac{y_{-1}}{x_{-1}}, \frac{y_{-1}^3}{x_{-1}^3}\right\} = \frac{y_{-1}^3}{x_{-1}^3} > \overline{y}$$

$$x_{12} = \max\left\{\frac{1}{x_9}, \frac{y_9}{x_9}\right\} = \left\{\frac{x_0^2}{y_0^2}, \frac{x_0^3}{y_0^3}\right\} = \frac{x_0^2}{y_0^2} < \overline{x}$$

$$y_{12} = \max\left\{\frac{1}{y_9}, \frac{x_9}{y_9}\right\} = \max\left\{\frac{y_0}{x_0}, \frac{y_0^3}{x_0^3}\right\} = \frac{y_0^3}{x_0^3} > \overline{y}$$

$$\vdots$$

Assume that the following hold for $n = k$

$$x_{6k+4}=\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+1)},\quad x_{6k+5}=\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+1)},\ x_{6k+6}=\left(\frac{x_{0}}{y_{0}}\right)^{f(2k+1)},$$

$$x_{6k+7}=\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+3)},\quad x_{6k+8}=\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+3)},\ x_{6k+9}=\left(\frac{y_{0}}{x_{0}}\right)^{f(2k+3)}$$

$$x_{6k+4}=\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+2)},\quad x_{6k+5}=\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+2)},\ x_{6k+6}=\left(\frac{y_{0}}{x_{0}}\right)^{f(2k+2)},$$

$$x_{6k+7}=\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+2)},\quad x_{6k+8}=\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+2)},\ x_{6k+9}=\left(\frac{x_{0}}{y_{0}}\right)^{f(2k+2)}$$

Now, we show that it is also true for $n = k + 1$

$$x_{6k+10}=\max\left\{\frac{1}{x_{6k+7}},\frac{y_{6k+7}}{x_{6k+7}}\right\}=\max\left\{\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+3)},\frac{\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+2)}}{\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+3)}}\right\}$$

$$=\max\left\{\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+3)},\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+2)}\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+3)}\right\}$$

$$=\max\left\{\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+3)},\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+4)}\right\}$$

$$=\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+3)},$$

$$y_{6k+10}=\max\left\{\frac{1}{y_{6k+7}},\frac{x_{6k+7}}{y_{6k+7}}\right\}=\max\left\{\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+2)},\frac{\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+3)}}{\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+2)}}\right\}$$

$$=\max\left\{\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+2)},\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+3)}\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+2)}\right\}$$

$$=\max\left\{\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+2)},\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+4)}\right\}$$

$$=\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+4)},$$

$$x_{6k+11} = \max\left\{\frac{1}{x_{6k+8}}, \frac{y_{6k+8}}{x_{6k+8}}\right\} = \max\left\{\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+3)}, \frac{\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+2)}}{\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+3)}}\right\}$$

$$= \max\left\{\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+3)}, \left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+2)}\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+3)}\right\}$$

$$= \max\left\{\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+3)}, \left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+4)}\right\}$$

$$= \left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+3)},$$

$$\mathrm{y}_{6k+11} = \max\left\{\frac{1}{y_{6k+8}}, \frac{x_{6k+8}}{y_{6k+8}}\right\} = \max\left\{\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+2)}, \frac{\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+3)}}{\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+2)}}\right\}$$

$$= \max\left\{\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+2)}, \left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+3)}\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+2)}\right\}$$

$$= \max\left\{\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+2)}, \left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+4)}\right\}$$

$$= \left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+4)},$$

$$x_{6k+12} = \max\left\{\frac{1}{x_{6k+9}}, \frac{y_{6k+9}}{x_{6k+9}}\right\} = \max\left\{\left(\frac{x_0}{y_0}\right)^{f(2k+2)}, \frac{\left(\frac{x_0}{y_0}\right)^{f(2k+1)}}{\left(\frac{y_0}{x_0}\right)^{f(2k+2)}}\right\}$$

$$= \max\left\{\left(\frac{x_0}{y_0}\right)^{f(2k+2)}, \left(\frac{x_0}{y_0}\right)^{f(2k+1)}\left(\frac{x_0}{y_0}\right)^{f(2k+2)}\right\}$$

$$= \max\left\{\left(\frac{x_0}{y_0}\right)^{f(2k+2)}, \left(\frac{x_0}{y_0}\right)^{f(2k+3)}\right\}$$

$$= \left(\frac{x_0}{y_0}\right)^{f(2k+3)},$$

$$
\begin{aligned}
\mathrm{y}_{6k+12} &= \max\left\{\frac{1}{y_{6k+9}},\frac{x_{6k+9}}{y_{6k+9}}\right\} = \max\left\{\left(\frac{y_0}{x_0}\right)^{f(2k+2)},\frac{\left(\frac{y_0}{x_0}\right)^{f(2k+2)}}{\left(\frac{x_0}{y_0}\right)^{f(2k+1)}}\right\} \\
&= \max\left\{\left(\frac{y_0}{x_0}\right)^{f(2k+2)},\left(\frac{y_0}{x_0}\right)^{f(2k+2)}\left(\frac{y_0}{x_0}\right)^{f(2k+1)}\right\} \\
&= \max\left\{\left(\frac{y_0}{x_0}\right)^{f(2k+2)},\left(\frac{y_0}{x_0}\right)^{f(2k+9)}\right\} \\
&= \left(\frac{y_0}{x_0}\right)^{f(2k+3)},
\end{aligned}
$$

$$
\begin{aligned}
x_{6k+12} &= \max\left\{\frac{1}{x_{6k+9}},\frac{y_{6k+9}}{x_{6k+9}}\right\} = \max\left\{\left(\frac{y_0}{x_0}\right)^{f(2k+2)},\frac{\left(\frac{x_0}{y_0}\right)^{f(2k+1)}}{\left(\frac{y_0}{x_0}\right)^{f(2k+2)}}\right\} \\
&= \max\left\{\left(\frac{y_0}{x_0}\right)^{f(2k+2)},\left(\frac{y_0}{x_0}\right)^{f(2k+1)}\left(\frac{x_0}{y_0}\right)^{f(2k+2)}\right\} \\
&= \max\left\{\left(\frac{y_0}{x_0}\right)^{f(2k+2)},\left(\frac{x_0}{y_0}\right)^{f(2k+3)}\right\} \\
&= \left(\frac{y_0}{x_0}\right)^{f(2k+2)},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{y}_{6k+12} &= \max\left\{\frac{1}{y_{6k+9}},\frac{x_{6k+9}}{y_{6k+9}}\right\} = \max\left\{\left(\frac{y_0}{x_0}\right)^{f(2k+1)},\frac{\left(\frac{y_0}{x_0}\right)^{f(2k+2)}}{\left(\frac{x_0}{y_0}\right)^{f(2k+1)}}\right\} \\
&= \max\left\{\left(\frac{y_0}{x_0}\right)^{f(2k+1)},\left(\frac{y_0}{x_0}\right)^{f(2k+2)}\left(\frac{y_0}{x_0}\right)^{f(2k+1)}\right\} \\
&= \max\left\{\left(\frac{y_0}{x_0}\right)^{f(2k+1)},\left(\frac{y_0}{x_0}\right)^{f(2k+3)}\right\} \\
&= \left(\frac{y_0}{x_0}\right)^{f(2k+9)},
\end{aligned}
$$

$$x_{6k+13} = \max\left\{\frac{1}{x_{6k+10}}, \frac{y_{6k+10}}{x_{6k+10}}\right\} = \max\left\{\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+2)}, \frac{\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+3)}}{\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+2)}}\right\}$$

$$= \max\left\{\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+2)}, \left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+3)}\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+4)}\right\}$$

$$= \max\left\{\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+2)}, \left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+4)}\right\}$$

$$= \left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+4)},$$

$$y_{6k+13} = \max\left\{\frac{1}{y_{6k+10}}, \frac{x_{6k+10}}{y_{6k+10}}\right\} = \max\left\{\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+3)}, \frac{\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+2)}}{\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2k+3)}}\right\}$$

$$= \max\left\{\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+3)}, \left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+2)}\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+3)}\right\}$$

$$= \max\left\{\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+3)}, \left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+4)}\right\}$$

$$= \left(\frac{x_{-2}}{y_{-2}}\right)^{f(2k+3)},$$

$$x_{6k+14} = \max\left\{\frac{1}{x_{6k+11}}, \frac{y_{6k+11}}{x_{6k+11}}\right\} = \max\left\{\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+2)}, \frac{\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+3)}}{\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+4)}}\right\}$$

$$= \max\left\{\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+2)}, \left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+3)}\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+2)}\right\}$$

$$= \max\left\{\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+2)}, \left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+4)}\right\}$$

$$= \left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+4)},$$

and

$$y_{6k+14}=\max\left\{\frac{1}{y_{6k+11}},\frac{x_{6k+11}}{y_{6k+11}}\right\}=\max\left\{\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+3)},\frac{\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+2)}}{\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2k+3)}}\right\}$$

$$=\max\left\{\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+3)},\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+2)}\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+3)}\right\}$$

$$=\max\left\{\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+3)},\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+4)}\right\}$$

$$=\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2k+3)}$$

$$x_{6k+15}=\max\left\{\frac{1}{x_{6k+12}},\frac{y_{6k+12}}{x_{6k+12}}\right\}=\max\left\{\left(\frac{y_{0}}{x_{0}}\right)^{f(2k+2)},\frac{\left(\frac{y_{0}}{x_{0}}\right)^{f(2k+3)}}{\left(\frac{x_{0}}{y_{0}}\right)^{f(2k+2)}}\right\}$$

$$=\max\left\{\left(\frac{y_{0}}{x_{0}}\right)^{f(2k+2)},\left(\frac{y_{0}}{x_{0}}\right)^{f(2k+3)}\left(\frac{y_{0}}{x_{0}}\right)^{f(2k+2)}\right\}$$

$$=\max\left\{\left(\frac{y_{0}}{x_{0}}\right)^{f(2k+2)},\left(\frac{x_{0}}{y_{0}}\right)^{f(2k+4)}\right\}$$

$$=\left(\frac{y_{0}}{x_{0}}\right)^{f(2k+4)},$$

$$y_{6k+15}=\max\left\{\frac{1}{y_{6k+12}},\frac{x_{6k+12}}{y_{6k+12}}\right\}=\max\left\{\left(\frac{x_{0}}{y_{0}}\right)^{f(2k+3)},\frac{\left(\frac{x_{0}}{y_{0}}\right)^{f(2k+2)}}{\left(\frac{y_{0}}{x_{0}}\right)^{f(2k+3)}}\right\}$$

$$=\max\left\{\left(\frac{x_{0}}{y_{0}}\right)^{f(2k+3)},\left(\frac{x_{0}}{y_{0}}\right)^{f(2k+2)}\left(\frac{x_{0}}{y_{0}}\right)^{f(2k+3)}\right\}$$

$$=\max\left\{\left(\frac{x_{0}}{y_{0}}\right)^{f(2k+3)},\left(\frac{x_{0}}{y_{0}}\right)^{f(2k+4)}\right\}$$

$$=\left(\frac{x_{0}}{y_{0}}\right)^{f(2k+3)},$$

is obtained. Thus, the validity of the above statements is established by mathematical induction.

According to the following initial conditions

$0 < y_{-2} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_0 < 1$, $0 < x_{-2} < x_{-1} < y_0 < x_0 < y_{-2} < y_{-1} < 1$,

$0 < y_{-2} < x_{-2} < x_{-1} < y_{-1} < y_0 < x_0 < 1$, $0 < x_{-2} < y_{-2} < y_{-1} < x_{-1} < x_0 < y_0 < 1$,

$0 < y_{-2} < x_{-2} < y_{-1} < x_{-1} < x_0 < y_0 < 1$, $0 < x_{-2} < y_{-2} < y_{-1} < x_{-1} < y_0 < x_0 < 1$,

$0 < y_{-2} < x_{-2} < y_{-1} < x_{-1} < y_0 < x_0 < 1$.

The proof for the solutions x_n and y_n is obtained similarly to the proof of Theorem 1.

Theorem 2: (1) System of equations,

$$0 < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-1} < y_0 < 1$$

Depending on the initial condition

a)
$$\begin{aligned}
&\lim_{n\to\infty} x_{6n+1} = \infty, \ \lim_{n\to\infty} x_{6n+2} = \infty, \ \lim_{n\to\infty} x_{6n+3} = \infty,\\
&\lim_{n\to\infty} x_{6n+4} = 0, \ \lim_{n\to\infty} x_{6n+5} = 0, \ \lim_{n\to\infty} x_{6n+6} = 0,\\
&\lim_{n\to\infty} x_{6n+7} = \infty, \ \lim_{n\to\infty} x_{6n+8} = \infty, \ \lim_{n\to\infty} x_{6n+9} = \infty
\end{aligned}$$

b)
$$\begin{aligned}
&\lim_{n\to\infty} y_{6n+1} = \infty, \ \lim_{n\to\infty} y_{6n+2} = \infty, \ \lim_{n\to\infty} y_{6n+3} = \infty,\\
&\lim_{n\to\infty} y_{6n+4} = \infty, \ \lim_{n\to\infty} y_{6n+5} = \infty, \ \lim_{n\to\infty} y_{6n+6} = \infty,\\
&\lim_{n\to\infty} y_{6n+7} = 0, \ \lim_{n\to\infty} y_{6n+8} = 0, \ \lim_{n\to\infty} y_{6n+9} = 0
\end{aligned}$$

Proof:

a) We obtain,

For $0 < x_{-2} < 1$, $\lim\limits_{n\to\infty} x_{6n+1} = \lim\limits_{n\to\infty}\left(\dfrac{1}{x_{-2}}\right) = \dfrac{1}{x_{-2}} = \infty$

For $0 < x_{-1} < 1$, $\lim\limits_{n\to\infty} x_{6n+2} = \lim\limits_{n\to\infty}\left(\dfrac{1}{x_{-1}}\right) = \dfrac{1}{x_{-1}} = \infty$

For $0 < x_0 < 1$, $\lim\limits_{n\to\infty} x_{6n+3} = \lim\limits_{n\to\infty}\left(\dfrac{1}{x_0}\right) = \dfrac{1}{x_0} = \infty$

For $0 < x_{-2} < y_{-2} < 1$, $\lim\limits_{n\to\infty} x_{6n+4} = \lim\limits_{n\to\infty}\left(\dfrac{x_{-2}}{y_{-2}}\right)^{f(2n+1)} = \left(\dfrac{x_{-2}}{y_{-2}}\right)^{f(\infty)} = \left(\dfrac{x_{-2}}{y_{-2}}\right)^{\infty} = 0$

For $0 < x_{-1} < y_{-1} < 1$, $\lim\limits_{n\to\infty} x_{6n+5} = \lim\limits_{n\to\infty}\left(\dfrac{x_{-1}}{y_{-1}}\right)^{f(2n+1)} = \left(\dfrac{x_{-1}}{y_{-1}}\right)^{f(\infty)} = \left(\dfrac{x_{-1}}{y_{-1}}\right)^{\infty} = 0$

For $0 < x_0 < y_0 < 1$, $\lim_{n\to\infty} x_{6n+6} = \lim_{n\to\infty}\left(\frac{x_0}{y_0}\right)^{f(2n+1)} = \left(\frac{x_0}{y_0}\right)^{f(\infty)} = \left(\frac{x_0}{y_0}\right)^{\infty} = 0$

For $0 < x_{-2} < y_{-2} < 1$, $\lim_{n\to\infty} x_{6n+7} = \lim_{n\to\infty}\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2n+3)} = \left(\frac{y_{-2}}{x_{-2}}\right)^{f(\infty)} = \left(\frac{y_{-2}}{x_{-2}}\right)^{\infty} = \infty$

For $0 < x_{-1} < y_{-1} < 1$, $\lim_{n\to\infty} x_{6n+8} = \lim_{n\to\infty}\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2n+3)} = \left(\frac{y_{-1}}{x_{-1}}\right)^{f(\infty)} = \left(\frac{y_{-1}}{x_{-1}}\right)^{\infty} = \infty$

For $0 < x_0 < y_0 < 1$, $\lim_{n\to\infty} x_{6n+9} = \lim_{n\to\infty}\left(\frac{y_0}{x_0}\right)^{f(2n+3)} = \left(\frac{y_0}{x_0}\right)^{f(\infty)} = \left(\frac{y_0}{x_0}\right)^{\infty} = \infty$

b) We obtain

For $0 < y_{-2} < 1$, $\lim_{n\to\infty} y_{6n+1} = \lim_{n\to\infty}\left(\frac{1}{y_{-2}}\right) = \frac{1}{y_{-2}} = \infty$

For $0 < y_{-1} < 1$, $\lim_{n\to\infty} y_{6n+2} = \lim_{n\to\infty}\left(\frac{1}{y_{-1}}\right) = \frac{1}{y_{-1}} = \infty$

For $0 < y_0 < 1$, $\lim_{n\to\infty} y_{6n+3} = \lim_{n\to\infty}\left(\frac{1}{y_0}\right) = \frac{1}{y_0} = \infty$

For $0 < x_{-2} < y_{-2} < 1$, $\lim_{n\to\infty} y_{6n+4} = \lim_{n\to\infty}\left(\frac{y_{-2}}{x_{-2}}\right)^{f(2n+2)} = \left(\frac{y_{-2}}{x_{-2}}\right)^{f(\infty)} = \left(\frac{y_{-2}}{x_{-2}}\right)^{\infty} = \infty$

For $0 < x_{-1} < y_{-1} < 1$, $\lim_{n\to\infty} y_{6n+5} = \lim_{n\to\infty}\left(\frac{y_{-1}}{x_{-1}}\right)^{f(2n+2)} = \left(\frac{y_{-1}}{x_{-1}}\right)^{f(\infty)} = \left(\frac{y_{-1}}{x_{-1}}\right)^{\infty} = \infty$

For $0 < x_0 < y_0 < 1$, $\lim_{n\to\infty} y_{6n+6} = \lim_{n\to\infty}\left(\frac{y_0}{x_0}\right)^{f(2n+2)} = \left(\frac{y_0}{x_0}\right)^{f(\infty)} = \left(\frac{y_0}{x_0}\right)^{\infty} = \infty$

For $0 < x_{-2} < y_{-2} < 1$, $\lim_{n\to\infty} y_{6n+7} = \lim_{n\to\infty}\left(\frac{x_{-2}}{y_{-2}}\right)^{f(2n+2)} = \left(\frac{x_{-2}}{y_{-2}}\right)^{f(\infty)} = \left(\frac{x_{-2}}{y_{-2}}\right)^{\infty} = 0$

For $0 < x_{-1} < y_{-1} < 1$, $\lim_{n\to\infty} y_{6n+8} = \lim_{n\to\infty}\left(\frac{x_{-1}}{y_{-1}}\right)^{f(2n+2)} = \left(\frac{x_{-1}}{y_{-1}}\right)^{f(\infty)} = \left(\frac{x_{-1}}{y_{-1}}\right)^{\infty} = 0$

For $0 < x_0 < y_0 < 1$, $\lim_{n\to\infty} y_{6n+9} = \lim_{n\to\infty}\left(\frac{x_0}{y_0}\right)^{f(2n+2)} = \left(\frac{x_0}{y_0}\right)^{f(\infty)} = \left(\frac{x_0}{y_0}\right)^{\infty} = 0$.

According to the following initial conditions

$0 < y_{-2} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_0 < 1$, $0 < x_{-2} < x_{-1} < y_0 < x_0 < y_{-2} < y_{-1} < 1$,

$0 < y_{-2} < x_{-2} < x_{-1} < y_{-1} < y_0 < x_0 < 1$, $0 < x_{-2} < y_{-2} < y_{-1} < x_{-1} < x_0 < y_0 < 1$,

$0 < y_{-2} < x_{-2} < y_{-1} < x_{-1} < x_0 < y_0 < 1$, $0 < x_{-2} < y_{-2} < y_{-1} < x_{-1} < y_0 < x_0 < 1$,

$0 < y_{-2} < x_{-2} < y_{-1} < x_{-1} < y_0 < x_0 < 1$

The proof for the solutions x_n and y_n is obtained similarly to the proof of Theorem 2.

EXAMPLE 1: If the initial conditions are selected as follows:

$x[-2]=0.2$; $x[-1]=0.3$; $x[0]=0.4$; $y[-2]=0.5$; $y[-1]=0.6$; $y[0]=0.7$;

then the following solutions are obtained:

x(n)={5.,3.3333333333333335,2.5,0.4,0.5,0.5714285714285715,6.25,4.,3.0624999999999996,0.16,0.25,0.326530612244898,97.65625,32.,16.413085937499996,0.01024,…}

y(n)={2.,1.6666666666666667,1.4285714285714286,2.5,2.,1.75,0.4,0.5,0.5714285714285714,15.625,8.,5.359374999999999,0.064,0.125,0.18658892128279886,1525.87890625,…}

The graph of the solutions is given below.

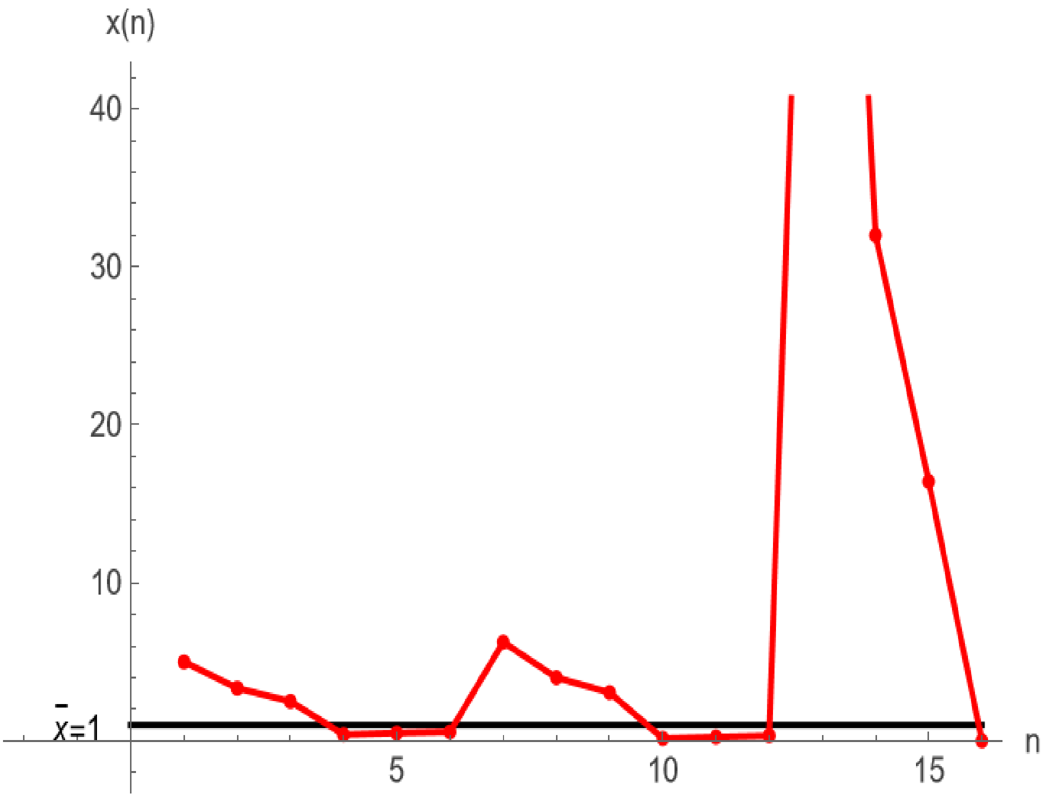

Figure 1. x(n) graph of the solutions

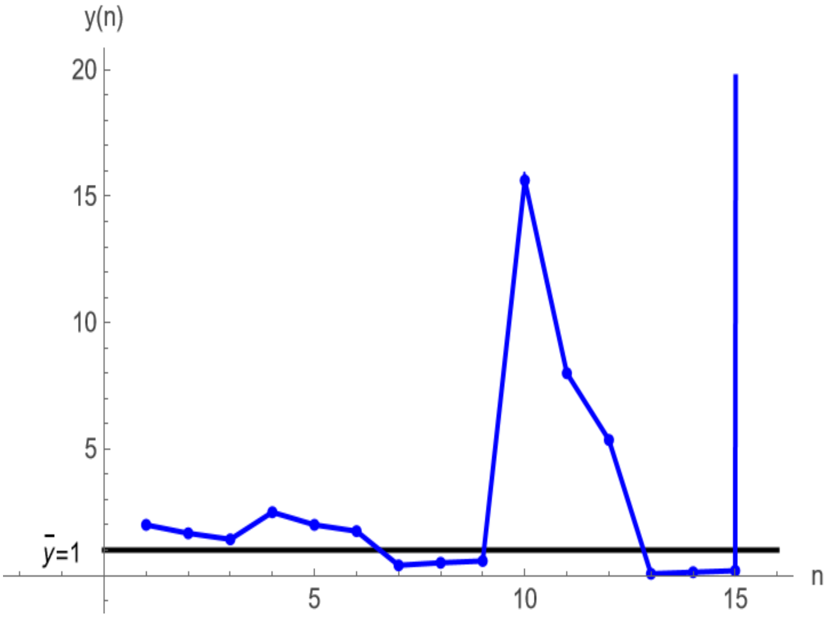

Figure 2. y(n) graph of the solutions

EXAMPLE 2: If the initial conditions are selected as follows:

$x[-2]$=0.02;$x[-1]$=0.03;$x[0]$=0.04;$y[-2]$=0.05;$y[-1]$=0.06;$y[0]$=0.07;

then the following solutions are obtained:

x(n)={50.,33.333333333333336,25.,0.4,0.5,0.5714285714285714,6.25,4.,3.0625000000000004,0.16,0.25,0.32653061224489793,97.65625,32.,16.413085937500004,0.01024,…}

y(n)={20.,16.666666666666668,14.285714285714285,2.5,2.,1.7500000000000002,0.4,0.5,0.5714285714285714,15.625,8.,5.359375000000001,0.064,0.125,0.1865889212827988,1525.87890625,…}

The graph of the solutions is given below.

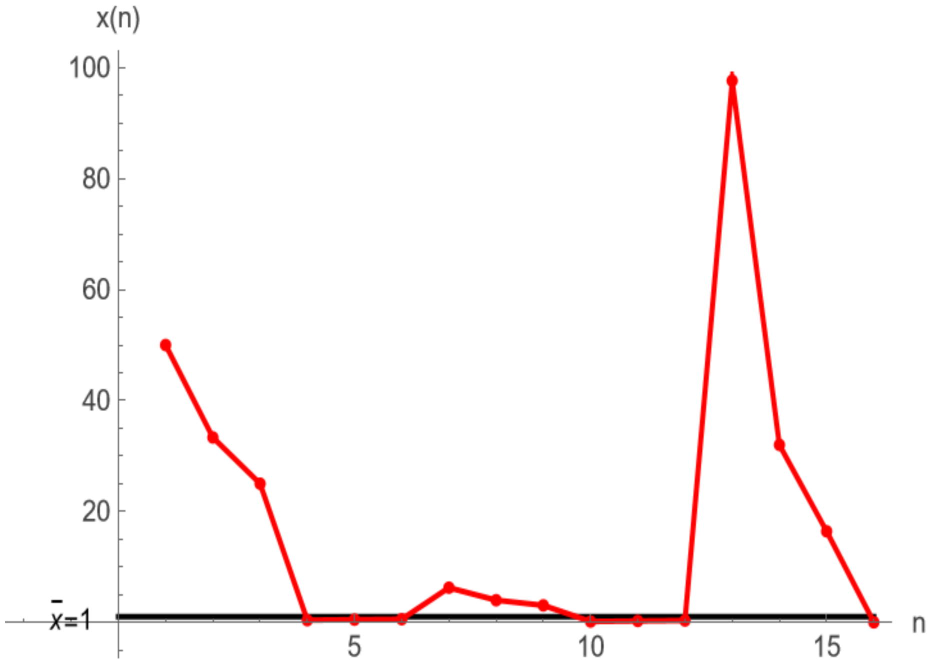

Figure 3. x(n) graph of the solutions

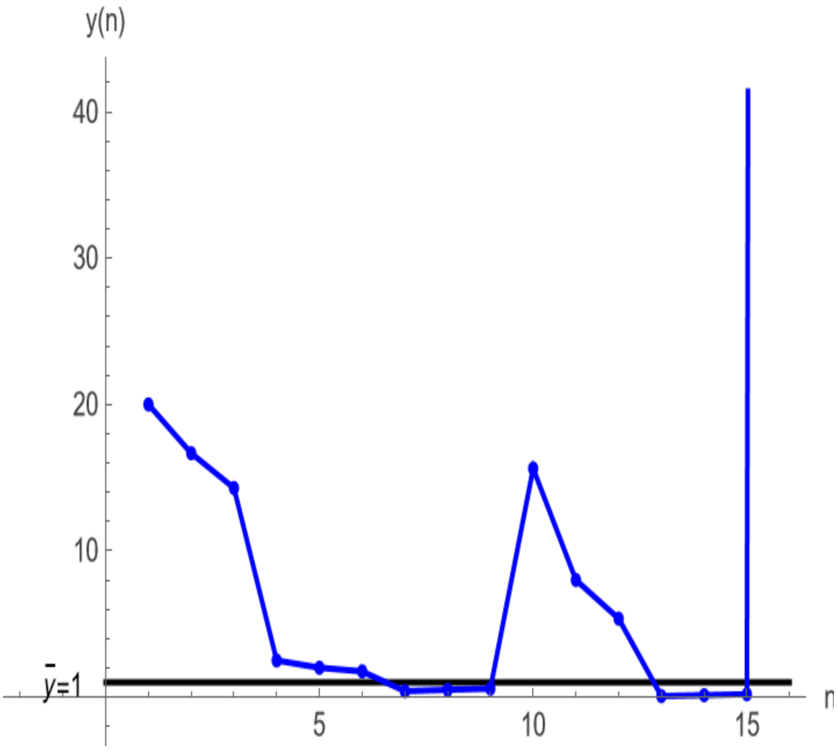

Figure 4. y(n) graph of the solutions

Conclusion

Given that the initial conditions x_{-2}, x_{-1}, x_0 are positive real numbers, $x_{n+1} = \max\left\{\frac{1}{x_{n-2}}, \frac{y_{n-2}}{x_{n-2}}\right\}$, $y_{n+1} = \max\left\{\frac{1}{y_{n-2}}, \frac{x_{n-2}}{y_{n-2}}\right\}$ the behavior of the solutions for the system of max-type difference equations has been investigated**.**

$$x_{n+1} = \max\left\{\frac{1}{x_{n-k}}, \frac{y_{n-k}}{x_{n-k}}\right\}, y_{n+1} = \max\left\{\frac{1}{y_{n-k}}, \frac{x_{n-k}}{y_{n-k}}\right\}$$

For future studies, researchers may investigate the solutions of the newly constructed system of difference equations, for various values of k.

Scientific Ethics Declaration

* The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

Conflict of Interest

* The authors declare that there is no conflict of interest

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