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On the Incompleted Sequences

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Abstract: In this study, Fibonacci, Lucas, and generalized forms of these sequences are examined. In the first chapter, definitions of Fibonacci and Lucas number sequences are provided, and some of their properties are analyzed. In the second chapter, incomplete Fibonacci and Lucas numbers are analyzed. Finally, in the third chapter, incomplete Fibonacci and Lucas numbers with arithmetic indices are defined and new relations are obtained

Keywords: Sequences with arithmetic indices, Fibonacci, Lucas, Incomplete Fibonacci and Lucas sequences

Introduction

Fibonacci Sequence

Let F_n denote the *n*. term of the Fibonacci sequence; for $n \geq 3$ under the initial conditions $F_1 = 1, F_2 = 1$ numbers satisfying this relation are called Fibonacci numbers (Koshy, 2001).

$$F_n = F_{n-1} + F_{n-2} \tag{1.1}$$

According to this definition, the first few Fibonacci numbers are as follows:

$F_0 = 0$

$F_1 = 1$

$F_2 = 1$

$F_3 = F_2 + F_1 = 1 + 1 = 2$

$F_4 = F_3 + F_2 = 2 + 1 = 3$

$F_5 = F_4 + F_3 = 3 + 2 = 5$

$F_6 = F_5 + F_4 = 5 + 3 = 8$

Binet Formula

Let $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$. Since each term of the Fibonacci sequence can be written as a linear combination of α and β, we obtain the equation

$$F_n = A\alpha^n + B\beta^n \tag{1.2}$$

When the initial conditions are substituted into this equation,

$F_0 = A + B = 0$

if the expression $B = -A$ is substitued,

$F_1 = A\alpha + B\beta = 1$

$= A\alpha + (-A)\beta = 0$

$= A(\alpha - \beta) = 1$

$A = \frac{1}{\alpha-\beta} \qquad B = -\frac{1}{\alpha-\beta}$

When the coefficients *A* and *B* are substituted into equation (1.2),

$$F_n = \left(\frac{1}{\alpha-\beta}\right)\alpha^n + \left(\frac{-1}{\alpha-\beta}\right)\beta^n$$

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

is obtained, and this equation we have found is called the Binet Formula (Koshy, 2001).

Some Properties of Fibonacci Numbers

In this section, we will examine a few of the most important of the many mathematical relationships presented by Fibonacci numbers. In particular, our next property will play a central role in the proofs of these relationships (Koshy, 2001). It is well-known that

$\sum_{k=0}^{n} x^k = \frac{1-x^{(n+1)}}{1-x}$ and $x < 1, n \to \infty$, $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.

- The sum of consecutive Fibonacci numbers is expressed as follows.

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1$$

Proof: Let us perform the proof using the Binet formula.

$$\sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \frac{\alpha^i - \beta^i}{\alpha - \beta}$$

$$= \frac{1}{\alpha - \beta}\left(\sum_{i=1}^{n} \alpha^i - \sum_{i=1}^{n} \beta^i\right)$$

$$= \frac{1}{\alpha - \beta}\left(\sum_{i=0}^{n-1} \alpha^{i+1} - \sum_{i=0}^{n-1} \beta^{i+1}\right)$$

$$= \frac{1}{\alpha - \beta}\left(\alpha \frac{1-\alpha^n}{1-\alpha} - \beta \frac{1-\beta^n}{1-\beta}\right)$$

When the inequalities $\alpha + \beta = 1, \alpha\beta = -1$ are used,

$$= \frac{1}{\alpha - \beta}\left(-\alpha^2(1-\alpha^n) + \beta^2 - (1-\beta^n)\right)$$

$$= \frac{1}{\alpha - \beta}\left(\alpha^{n+2} - \beta^{n+2} - (\alpha^2 - \beta^2)\right)$$

$$= \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} - \frac{(\alpha - \beta)(\alpha + \beta)}{\alpha - \beta}$$

$$= F_{n+2} - 1$$

- For every integer $n \geq 1$ the following equations are obtained (Koshy, 2001).

$$\sum_{i=1}^{n} F_{2i-1} = F_{2n}$$

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$$

Proof:

$$\sum_{i=1}^{n} F_{2i-1} = \sum_{i=1}^{n} \frac{\alpha^{2i-1} - \beta^{2i-1}}{\alpha - \beta}$$

$$= \frac{1}{\alpha - \beta}\left(\alpha \sum_{i=1}^{n} \alpha^{2i} - \beta \sum_{i=1}^{n} \beta^{2i}\right)$$

$$= \frac{1}{\alpha - \beta}\left(\alpha \frac{1-\alpha^{2n}}{(1-\alpha)(1-\beta)} - \beta \frac{1-\beta^{2n}}{(1-\alpha)(1-\beta)}\right)$$

$$= \frac{1}{\alpha - \beta}\left(-\alpha^2 \frac{1-\alpha^{2n}}{1+\alpha} + \beta^2 \frac{1-\beta^{2n}}{1+\beta}\right)$$

$$= \frac{1}{\alpha - \beta}(-1 + \alpha^{2n} + 1 - \beta^{2n})$$

$$= \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}$$

- For every natural number $n \geq 1$ two consecutive Fibonacci numbers are coprime (relatively prime) (Koshy, 2001).

$$(F_{n+1}, F_n) = 1$$

Proof: By induction. It is obvious that, for n=1,

$$(F_2, F_1) = (1{,}1) = 1$$

Assume it is true for n=k. Namely,

$$(F_{k+1}, F_k) = 1.$$

Is it true for n=k+1 ? Let $(F_{k+2}, F_{k+1}) = d$. Since $d\ |F_{k+2}$, $d|F_{k+1}$ and $F_k = F_{k+2} - F_{k+1}$ it follows that $d \mid (F_{k+2} - F_{k+1})$. Thus, $d \mid F_k$ is obtained. $d \leq (F_{k+1}, F_k)$, that is, $d \leq 1$.Since $d \geq 1$ by the definition of the GCD $d = 1$ is obtained. $(F_{k+2}, F_{k+1}) = 1$ is found.

- For every natural number $n \geq 1$(Koshy, 2001).

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

Proof: Let us solve it using the method of induction.

$$n = 1$$

$$F_0 F_2 - F_1 = 0 * 1 - 1^2 = (-1)^2$$

Assume it is true for *n=k.*

$$F_{k-1}F_{k+1} - F_k^2 = (-1)^k$$

Is it true for *n=k+1* ?

$$F_k F_{k+2} - F_{k+1}^2 = (-1)^{(k+1)}$$

$$= (F_{k+1} - F_{k-1})(F_k + F_{k+1}) - F_{k+1}^2$$

$$= F_{k+1}F_k + F_{k+1}^2 - F_{k-1}F_k - F_{k-1}F_{k+1} - F_{k+1}^2$$

$= F_{k+1}F_k - F_{k-1}F_k - F_k^2 - (-1)^k$

$= F_kF_{k+1} - F_k(F_{k-1} + F_k) + (-1)^{(k+1)}$

$= F_kF_{k+1} - F_kF_{k+1} + (-1)^{(k+1)}$

$= (-1)^{(k+1)}$

follows as claimed.

Lucas (Companıon) Sequence

When Fibonacci numbers are mentioned, another closely related important number sequence that comes to mind is the Lucas numbers. The terms of this sequence are denoted by L_n. Lucas numbers use exactly the same recurrence relation as Fibonacci numbers, but they have different initial conditions (Koshy, 2001).

With the initial conditions being $L_0 = 2$, $L_1 = 1$ a term is the sum of the two preceding terms. That is, for $n \geq 2$,

$$L_n = L_{n-1} + L_{n-2} \tag{1.3)}$$

According to this definition, the first few Lucas numbers are as follows:

$L_0 = 2$

$L_1 = 1$

$L_2 = 3$

$L_3 = L_2 + L_1 = 3 + 1 = 4$

$L_4 = L_3 + L_2 = 4 + 3 = 7$

$L_5 = L_4 + L_3 = 7 + 4 = 11$

$L_6 = L_5 + L_4 = 11 + 7 = 18$

$\vdots$

Just as Fibonacci numbers, there is a Binet formula for Lucas numbers that serves to calculate the *n.* term directly. Given the golden ratio $\alpha = \frac{1+\sqrt{5}}{2}$ known from the Fibonacci formula and its conjugate $\beta = \frac{1-\sqrt{5}}{2}$, the Binet formula for Lucas numbers is in the form of $L_n = \alpha^n + \beta^n$.

There are many identities between the Fibonacci and Lucas number sequences. Some of the most fundamental ones are the following:

- Let *n* be a natural number, $L_nF_n = F_{2n}$ (Koshy, 2001).

Proof:

$$L_nF_n = (\alpha^n + \beta^n)\left(\frac{\alpha^n - \beta^n}{\sqrt{5}}\right)$$

$$= \frac{(\alpha^n)^2 - (\beta^n)^2}{\sqrt{5}}$$

$$= \frac{\alpha^{2n} - \beta^{2n}}{\sqrt{5}}$$

$$= F_{2n}$$

- For every natural number $n \geq 1$, $L_n = F_{n+1} + F_{n-1}$ is obtained.

Proof:

$$F_{n+1} + F_{n-1} = \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{5}}\right) + \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}}\right)$$

$$= \frac{1}{\sqrt{5}}[(\alpha^{n+1}\alpha^{n-1}) - (\beta^{n+1} + \beta^{n-1})]$$

$$= \frac{1}{\sqrt{5}}[\alpha^n(\alpha + \alpha^{-1}) - \beta^n(\beta + \beta^{-1})]$$

$$= \frac{1}{\sqrt{5}}[\alpha^n\sqrt{5} - \beta^n\sqrt{5}]$$

$$= \frac{\sqrt{5}}{\sqrt{5}}(\alpha^n + \beta^n)$$

$$= \alpha^n + \beta^n = L_n$$

Fibonacci Generalizations

Theoretical research on Fibonacci and Lucas sequences has shown that these sequences are special examples of a broader family of second-order linear recurrence sequences. Studies in this field generally focus on generalizing the structure of these sequences (Swamy, 1999). The most comprehensive and widely studied of these generalizations is the approach that preserves the second-order structure of the sequence and extends both the initial conditions and the coefficients in the recurrence relation to arbitrary integers. This general sequence is known as the Horadam Sequence in reference to the pioneering works ofHoradam (Belbachir & Amine, 2018). It is denoted by (U_n).

Let p and q be distinct integer coefficients, and a and b be arbitrary integer initial conditions. The Horadam sequence $(U_n)$is defined by a linear recurrence relation for $n > 1$. The first two terms of the sequence are $U_0 = a$ and $U_1 = b$.

$$U_n = pU_{n-1} + qU_{n-2}$$

Now let us obtain the Binet formula of the sequence.

$$U_n = A\alpha^n + B\beta^n$$

$$U_0 = A + B = a$$

$U_1 = A\alpha + B\beta = b$ if the expression $B = a - A$ is substituted,

$$A\alpha + (a - A)\beta = b$$

$$A\alpha + a\beta - A\beta = b$$

$$A(\alpha - \beta) + \alpha\beta = b$$

$A = \frac{b - \alpha\beta}{\alpha - \beta}$ and $B = \frac{a\alpha - b}{\alpha - \beta}$

If we substitute the values of A and B we found into equation (1.4), we obtain the following Binet formula.

$$U_n = \left(\frac{b - \alpha\beta}{\alpha - \beta}\right)\alpha^n + \left(\frac{a\alpha - b}{\alpha - \beta}\right)\beta^n$$

Now let us obtain the generating function of the sequence.

$$h(x) = \sum_{n=0}^{\infty} U_n x^n \to h(x) = U_0 + U_1 x + U_2 x^2 + \cdots + U_n x^n + \cdots$$

$$h(x) = a + bx + \sum_{n=2}^{\infty} U_n x^n$$

$$h(x) = a + bx + px \sum_{n=2}^{\infty} (pU_{n-1} + qU_{n-2}) x^2$$

$$h(x) = a + bx + px \sum_{n=2}^{\infty} U_{n-1} x^{n-1} + qx^2 \sum_{n=2}^{\infty} U_{n-2} x^{n-2}$$

$$h(x) = a + bx + px(h(x) - a) + qx^2 h(x)$$

$$h(x) = a + bx + pxH(x) - apx + qx^2 h(x)$$

$$h(x) - pxh(x) - qx^2 h(x) = a + bx + pax$$

$$h(x)(1 - px - qx^2) = a + x(b - pa)$$

$$h(x) = \frac{a + x(b - pa)}{1 - px - qx^2}$$

if a and b are substituted;

$$h(x) = \frac{U_0 + x(U_1 - pU_0)}{1 - px - qx^2}$$

This definition unites many important sequences in number theory under a single roof with different integers assigned to the values of p, q, a, b. This unified approach allows for the investigation of structures such as Binet formulas and generating functions of different sequences (Belbachir & Amine, 2018; Koshy, 2001).

Table 1. Special number sequences

p	q	a	b	Notation	Name of the sequence
1	1	0	1	F_n	Fibonacci Sequence
1	1	2	1	L_n	Lucas Sequence
2	1	0	1	P_n	Pell Sequence
2	1	2	2	p_n	Pell-Lucas Sequence
1	2	0	1	J_n	Jakobsthal Sequence
1	2	2	1	j_n	Jakobsthal-Lucas Sequence

Incomplete Fibonacci and Lucas Numbers

Fibonacci and Lucas numbers are expressed through combinatorial sums (Koshy, 2001).

$$F_n = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-i}{i}$$

$$L_n = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-i} \binom{n-i}{i}$$

Incomplete Fibonacci Numbers

Many generalizations have been made in the literature for these special number sequences. One of these, the Incomplete Fibonacci numbers, is defined as follows (Filipponi, 1996);

$$F_n(k) = \sum_{j=0}^{k} \binom{n-1-j}{j}, \qquad 0 \le k \le \tilde{n}\,;\ \tilde{n} = \left\lfloor \frac{n-1}{2} \right\rfloor \tag{2.1}$$

The numbers $F_n(k)$ for a few values of n and the corresponding values of k are shown in Table 2.

Some special cases obtained from equation (2.1) are as follows.

- Let $n \ge 1$ and n be a natural number, $F_n(\tilde{n}) = F_n$
- For $n \ge 3$, $F_n(\tilde{n} - 1) = \begin{cases} F_n - \frac{n}{2}, & n \text{ is even} \\ F_n - 1, & n \text{ is odd} \end{cases}$
- For $n \ge 1$, $F_n(0) = 1$

Table 2. $F_n(k), 1 \le n \le 8$

n \ k	0	1	2	3	…
1	1				
2	1				
3	1	2			
4	1	3			
5	1	4	5		
6	1	5	8		
7	1	6	12	13	
8	1	7	17	21	
⋮	⋮	⋮	⋮	⋮	⋱

Proof: Let us give the proof of the 2nd property.

$$\sum_{j=0}^{\tilde{n}} \binom{n-1-j}{j} - \binom{n-1-\tilde{n}}{\tilde{n}}$$

$$= F_n - \binom{n-1-\tilde{n}}{\tilde{n}}$$

if $\rightarrow n = 2k \qquad (\tilde{n} = k-1)$

$$= F_n - \binom{2k-1-(k-1)}{k-1} = F_n - \binom{k}{k-1} = F_n - \frac{n}{2}$$

if $\rightarrow n = 2k+1 \qquad (\tilde{n} = k)$

$$= F_n - \binom{2k+1-1-k}{k} = F_n - \binom{k}{k} = F_n - 1$$

Incomplete Lucas Numbers

$$L_n(k) = \sum_{j=0}^{k} \frac{n}{n-j}\binom{n-j}{j} \qquad 0 \le k \le \tilde{n};\ \tilde{n} = \left\lfloor \frac{n}{2} \right\rfloor \tag{2.2}$$

The numbers $L_n(k)$ for a few values of *n* and the corresponding values of *k* are shown in Table 3 (Filipponi, 1996).

Some special cases obtained from equation (2.2) are given below.

- For $n \ge 1,\ L_n(0) = 1$
- For $n \ge 2,\ L_n(1) = n + 1$
- For $n \ge 4$, $L_n(2) = \frac{n^2-n+2}{2}$

Table 3. $L_n(k), 1 \leq n \leq 8$

k / n	0	1	2	3	…
1	1				
2	1	3			
3	1	4			
4	1	5	7		
5	1	6	11		
6	1	7	16	18	
7	1	8	22	29	
8	1	9	29	45	
⋮	⋮	⋮	⋮	⋮	⋱

Proof: Let us give the proof of the 3th property.

$$L_n(2) = \sum_{j=0}^{2} \frac{n}{n-j}\binom{n-j}{j}$$

$$= \frac{n}{n}\binom{n}{0} + \frac{n}{n-1}\binom{n-1}{1} + \frac{n}{n-2}\binom{n-2}{2}$$

$$= 1 + n + \frac{n}{n-2}\frac{(n-2)(n-3)}{2} = \frac{n^2-n+2}{2}$$

Incomplete Fibonacci and Lucas Numbers with Arithmetic Indices

Waring Formula

The Waring formula is a fundamental method ued in obtaining the explicit form of linear recurrences. This formula allows expressing the general term of the sequence directly, without resorting to the recursive definition, with the help of the roots of the characteristic equation. In particular, it can be considered as a generalized form of the known Binet formula for Fibonacci and Lucas-like sequences. The Waring formula for Fibonacci and Lucas numbers is as follows (Swamy,1999; Ramirez, 2013).

$$\frac{X^n - Y^n}{X - Y} = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k \binom{n-k-1}{k} (XY)^k (X+Y)^{n-2k-1} \tag{3.1}$$

$$X^n + Y^n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (XY)^k (X+Y)^{n-2k} \tag{3.2}$$

Incomplete Fibonacci Number Sequence with Arithmetic Indices

The Fibonacci number sequence with arithmetic indices is defined as follows (Kılıç & Stanica, 2009)

$$F_{rn} = L_r F_{r(n-1)} - (-1)^r F_{r(n-2)} \tag{3.3}$$

The Binet formula of the Fibonacci number sequence with arithmetic indices is obtained as follows.

$$F_{rn} = \frac{\alpha^{rn} - \beta^{rn}}{\alpha - \beta}$$

The explicit formula of the relation number (3.3) is obtained using the Waring Formula (3.1). This approach allowed for the systematic derivation of the explicit formula of the sequence. If the expressions

$X = \alpha^r, Y = \beta^r$ are substituted,

$$X + Y = \alpha^r + \beta^r = L_r$$

$$XY = \alpha^r \beta^r = (-1)^r$$

$$\frac{\alpha^{rn} - \beta^{rn}}{\alpha^r - \beta^r} = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k \binom{n-k-1}{k} ((-1)^r)^k (L_r)^{n-2k-1}$$

$$\frac{F_{rn}}{F_r} = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^{k(r+1)} \binom{n-k-1}{k} (L_r)^{n-2k-1}$$

$$F_{rn} = F_r \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^{k(r+1)} \binom{n-k-1}{k} (L_r)^{n-2k-1}$$

$$F_{rn}^l = F_r \sum_{k=0}^{l} (-1)^{k(r+1)} \binom{n-k-1}{k} (L_r)^{n-2k-1} \tag{3.4}$$

Table 4. F_{rn}^l, $r = 3, 1 \le n \le 8$

l n	0	1	2	3	...
1	2				
2	8				
3	32	34			
4	128	144			
5	512	608	610		
6	2048	2560	2584		
7	8192	10752	10944		
8	32768	45056	46336	46368	
⋮	⋮	⋮	⋮	⋮	⋱

Some special cases can be obtained for (3.4).

- Let $n \geq 1$ and n be a natural number, $F_{rn}^{\left\lfloor \frac{n-1}{2} \right\rfloor} = F_{rn}$
- For $n \geq 3$, $F_{rn}^{1} = F_r L_r^{n-1} + F_r(n-2)(-1)^{r+1} L_r^{n-3}$
- For $n \geq 5$, $F_{rn}^{2} = F_r L_r^{n-1} + F_r(n-2)(-1)^{r+1} L_r^{n-3} + F_r \frac{(n-3)(n-4)}{2} L_r^{n-5}$
- For $n \geq 3$, $F_{rn}^{\left\lfloor \frac{n-3}{2} \right\rfloor} = \begin{cases} F_{rn} - F_r(-1)^{\frac{n-1}{2}(r+1)}, n \text{ is odd} \\ F_{rn} - F_r \frac{n}{2} L_r(-1)^{\frac{n-2}{2}(r+1)}, n \text{ is even} \end{cases}$

Incomplete Lucas Numbers Sequence with Arithmetic Indices

The Lucas number sequence with arithmetic indices is defined as follows.

$$L_{rn} = L_r L_{r(n-1)} - (-1)^r L_{r(n-2)} \tag{3.5}$$

The Binet formula of the Lucas number sequence with arithmetic indices is obtained as follows.

$$L_{rn} = \alpha^{rn} + \beta^{rn}$$

The explicit formula of the relation number (3.5) is obtained using the Waring Formula (3.2). This approach allowed for the systematic derivation of the explicit formula of the sequence.

If the expressions $X = \alpha^r, Y = \beta^r$ are substituted,

$$X + Y = \alpha^r + \beta^r = L_r$$

$$XY = \alpha^r \beta^r = (-1)^r$$

$$\alpha^{rn} + \beta^{rn} = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} ((-1)^r)^k (L_r)^{n-2k}$$

$$L_{rn} = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^{k(r+1)} \frac{n}{n-k} \binom{n-k}{k} (L_r)^{n-2k}$$

$$L_{rn}^{l} = \sum_{k=0}^{l} (-1)^{k(r+1)} \frac{n}{n-k} \binom{n-k}{k} (L_r)^{n-2k} \tag{3.6}$$

Some special cases can be obtained for (3.6).

- Let $n \geq 1$ and n be a natural number, $L_{rn}^{\left\lfloor \frac{n-1}{2} \right\rfloor} = L_{rn}$
- For $n \geq 1$, $L_{rn}^{0} = (L_r)^n$
- For $n \geq 2$,$L_{rn}^{1} = L_r^n + n(-1)^{r+1}(L_r)^{n-2}$
- For $n \geq 4$, $L_{rn}^{2} = (L_r)^n + n(-1)^{r+1}(L_r)^{n-2} + \frac{n(n-3)}{2}(L_r)^{n-4}$

Table 5. L_{rn}^{l}, $r = 5, 1 \leq n \leq$

l / n	0	1	2	3	…
1	11				
2	121	123			
3	1331	1364			
4	14641	15125	15127		
5	161051	167706	167761		
6	1771561	1859407	1860496	1860498	
7	19487171	20614528	20633162	20633239	
8	214358881	228531369	228824189	228826125	
⋮	⋮	⋮	⋮	⋮	⋱

For $n \geq 3$, $L_{rn}^{\left\lfloor \frac{n-3}{2} \right\rfloor} = \begin{cases} L_{rn} - n(-1)^{\frac{n-1}{2}(r+1)} L_r, n \text{ is odd} \\ L_{rn} - \frac{n^2}{4}(-1)^{\left(\frac{n}{2}-1\right)(r+1)}(L_r)^2 + 2(-1)^{\frac{n}{2}(r+1)}, n \text{ is even} \end{cases}$

Method

In this study, various analytical and algebraic methods were employed to investigate the properties of Fibonacci and Lucas number sequence38s and to generalize these. The Binet formula, derived from the roots of the characteristic equation α and β was utilized to obtain the general terms of the fundamental sequences, the identities among them, and their generating functions. Furthermore, utilizing A.F. Horadam's approach to generalize second-order linear recurrence sequences, the generating functions of various sequences (e.g., Pell, Jacobsthal) were examined within a single unified framework (Belbachir & Amine, 2018; Koshy, 2001). In order to obtain the explicit formulas for the Incomplete Fibonacci and Lucas numbers, as well as their arithmetic-indexed versions discussed in the later stages of the study, the Waring formula—a highly effective tool for solving linear recurrences—was employed rather than relying on recursive definitions. The validity of the relevant theorems and identities was demonstrated through step-by-step proofs using algebraic manipulations and combinatorial sums.

Results and Discussion

As a result of this study, new analytical (closed-form) formulas were obtained for the arithmetic-indexed versions of the Incomplete Fibonacci and Lucas sequences. Through the effective utilization of Waring's formula, the general terms of the sequences could be computed directly, eliminating the computational burden introduced by recursive calculations. It has been algebraically proven that these general equations seamlessly reduce to the classical Fibonacci and Lucas sequences under special conditions. Moreover, by employing A.F. Horadam's approach, it was demonstrated that distinct sequences, such as Pell and Jacobsthal, can be unified under a single mathematical framework. It is anticipated that this developed analytical framework and the novel identities will establish a robust foundation for future research on higher-order number sequences.

Conclusion

In this study, new analytical formulas for the arithmetic-indexed versions of Incomplete Fibonacci and Lucas sequences were successfully obtained using Waring's formula. These results enabled the direct calculation of the general terms of the sequences without the need for recursive computations, unifying various number sequences under a common framework. It is expected that these newly introduced identities will lay a significant groundwork for future research on higher-order sequences.

Recommendations

For future research, it is recommended to extend the method developed in this study to third-order or higher-order number sequences (e.g., the Tribonacci sequence).

Scientific Ethics Declaration

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Conflict of Interest

* The authors declare that they have no conflicts of interest.

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