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Lengthwise Fracture Analysis of a Moving Thin-Walled Elliptical Cylinder

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Abstract: This paper is devoted to the problem of lengthwise fracture in moving thin-walled elliptical cylindrical shell structures with non-linear elastic behavior. Elliptical cylinders have continuous material inhomogeneity in height. The parameters of the non-linear constitutive law and the bulk density of the material of the cylinder wall change continuously in height. The lengthwise crack represents an elliptical surface whose location coincides with the middle surface of the cylinder. Two types of cylinder motion are considered (translational motion with constant upward acceleration and uniform rotation around the longitudinal axis). The cylinders are loaded with their own weight and with inertial forces from both types of motion. Strain energy release rate (SERR) solutions are obtained for both cylinder motions. In this regard, expressions for the complementary strain energy stored in both arms of the lengthwise crack and in the intact portion of the elliptical cylinder are obtained. A check was made with an existing approach in scientific literature.

Keywords: Elliptical cylinder, Motion, Lengthwise fracture

Introduction

Modern facilities, structures, mechanisms and machines include a variety of structural systems and components such as rods, plates, beams, thin-walled elements, shells, tanks, etc. with a variety of shapes, loads and support conditions (Kolev & Mladenov, 1992). The analysis of these structural systems and components requires solving a fairly wide range of diverse problems in the field of mechanics of rod systems, theory of elasticity and plasticity, theory of thin-walled systems, mechanics of continuous media, theory of shells and others (Malenov, 1993). Due to the complexity of the analyzed phenomena and processes in structural systems, it is very often necessary to introduce various simplifying hypotheses in practice.

In various fields of technology, thin-walled shell structures are increasingly used in our time. These structures are preferred primarily because of their low self-weight. The widespread use of these structures poses a number of specific tasks in the field of analysis of their behavior. In this regard, it can be noted that taking into account such factors as physical non-linearity, i.e. non-linear elastic or elastic-plastic behavior of various shell structural systems, material inhomogeneity, dynamic action of external load and others is of particular interest and introduces additional requirements for the analysis of shell structures (Mladenov et al., 2012; Obodovski & Hanin, 1971).

This paper deals with lengthwise fractured analysis of a moving thin walled continuously inhomogeneous shell. The latter is an elliptical cylinder with continuous material inhomogeneity along the length. The interest in the analysis of lengthwise cracks is due to the fact that very often continuously inhomogeneous materials, such as functionally graded ones, are produced by layering, which creates a risk of lengthwise fracture (Gandra et al., 2011; Gasik, 2010; Mahamood & Akinlabi, 2017; Rizov & Altenbach, 2019; El-Galy et al., 2019). The cylinder studied in this paper has non-linear elastic behavior. The SERR was obtained for two types of cylinder motion, namely translational motion with constant upward acceleration and uniform rotation around the longitudinal axis. The need for analysis of moving cylinders is due to their widespread use in various machines, mechanisms and vehicles. An existing approach to finding the SERR was used for verification.

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Analysis

The thin-walled elliptical cylinder in Fig. 1 has height, h , and wall thickness, b . The base of the cylinder is an ellipse with semi-major and semi-minor axes, r_1 and r_2 , respectively (Fig. 2). There is a lengthwise crack in the cylinder with length, a , as shown in Fig. 1. The crack is an elliptical cylindrical surface coinciding with the middle surface of the thin-walled elliptical cylinder (Fig. 1). Therefore, the outer and inner arms of the crack are elliptical cylinders with wall thickness, $b/2$, and height, a .

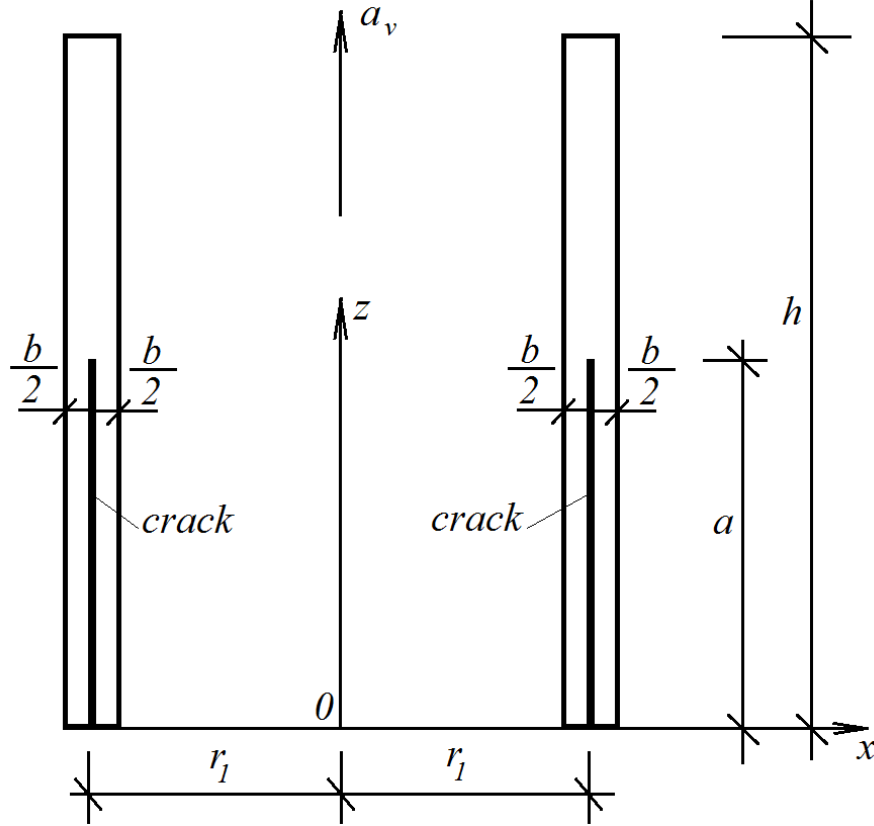


Figure 1. Thin-walled elliptical cylinder moving along the axis, z

First, we will analyze the case when the cylinder moves along the axis, z , with a constant acceleration, a_v , as shown in Fig. 1. The load acting on the thin-walled elliptical cylinder has two components, namely its own weight and the inertial load from the acceleration, a_v . Both components are directed vertically downward. We are looking for the meridional stress, σ_{ml} , in an arbitrary section in the outer arm of the crack at a distance, z , from the base of the thin-walled elliptical cylinder. For this purpose, we use the equilibrium condition

$$\sum F_{iz} = 0. \quad (1)$$

After substituting the force projections in (1) we obtain

$$\sigma_{ml} \frac{b}{2} c - z \gamma g \frac{b}{2} c - \gamma \frac{b}{2} z c a_v = 0, \quad (2)$$

where γ is the bulk density of the cylinder material, g is the acceleration due to gravity, c is the perimeter of the ellipse. The third term on the left-hand side of equation (2) is the inertial load caused by the acceleration, a_v .

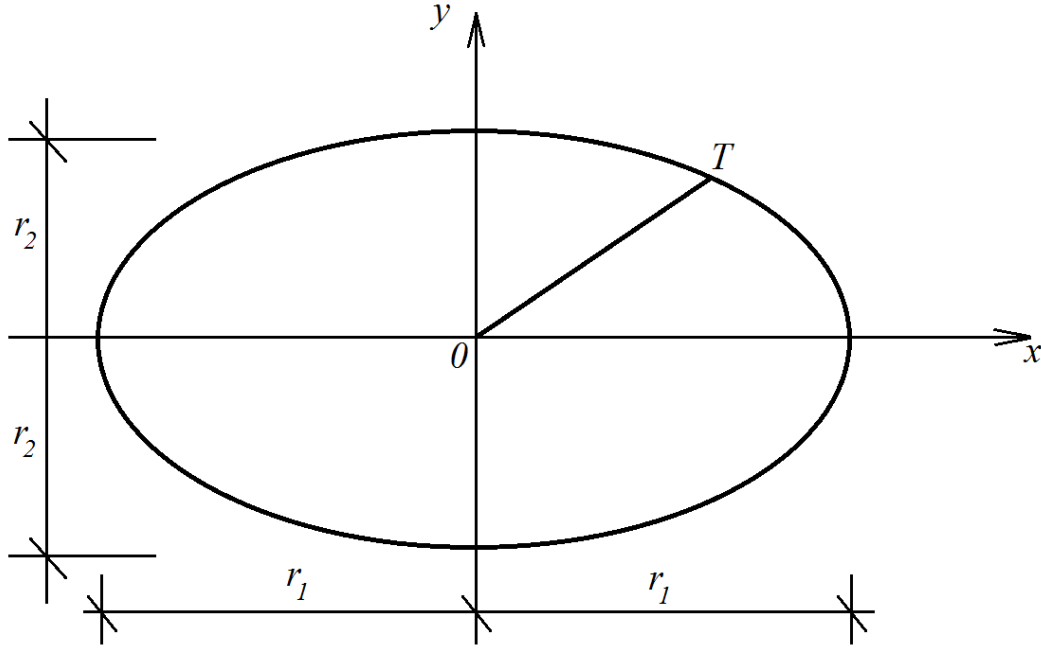


Figure 2. Schematic of the cross-section of the elliptical cylinder

The cylinder wall has a continuous material inhomogeneity in height. Therefore, the bulk density of the cylinder material varies continuously along the axis, z . To describe this variation, the following law is applied:

$$\gamma = \gamma_{\alpha} + \frac{\gamma_{\beta} - \gamma_{\alpha}}{h^{\delta}} z^{\delta}, \quad (3)$$

$$0 \leq z \leq h, \quad (4)$$

where γ_{α} and γ_{β} are the bulk densities at the lower and upper end of the cylinder, δ is a parameter. From equation (1) we obtain the following expression for the meridional stress in an arbitrary section in the outer arm of the crack

$$\sigma_{ml} = \gamma z(g + a_v), \quad (5)$$

where

$$0 \leq z \leq a. \quad (6)$$

The meridional stress, σ_{md} , in the inner arm of the crack is also found by formula (5).

It remains to determine the meridional stress, σ_m , in the intact portion, $z > a$, of the thin-walled elliptical cylinder. We formulate the condition for equilibrium along the axis, z

$$\sigma_m bc - z \gamma b c - \gamma b z c a_v = 0, \quad (7)$$

from which we obtain

$$\sigma_m = \gamma z(g + a_v), \quad (8)$$

Where

$$a \leq z \leq h. \quad (9)$$

We will prove that the hoop stress, σ_{pl} , in the outer arm of the crack is zero. To do this, we write the Laplace equation for the outer arm

$$\frac{\sigma_{ml}}{\rho_m} + \frac{\sigma_{pl}}{\rho_p} = \frac{q}{\frac{b}{2}}, \quad (10)$$

where ρ_m and ρ_p are the radii of curvature of the elliptical cylinder. The equation of the base of the thin-walled elliptical cylinder is (Fig. 2)

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1, \quad (11)$$

Where

$$-r_1 \leq x \leq r_1, \quad (12)$$

$$-r_2 \leq y \leq r_2. \quad (13)$$

Therefore, the radii of curvature of the elliptical cylinder are found by the formulas

$$\rho_m = \infty, \quad (14)$$

$$\rho_p = \frac{(1 + \theta_\tau^2)^{\frac{3}{2}}}{|\theta_\zeta|}, \quad (15)$$

Where

$$\theta_\tau = -\frac{r_2 x}{r_1^2 \sqrt{\theta_\kappa}}, \quad (16)$$

$$\theta_\zeta = -\frac{r_2}{r_1^2} \frac{r_1^2 \theta_\kappa + x^2}{r_1^2 \theta_\kappa^{\frac{3}{2}}}, \quad (17)$$

$$\theta_\kappa = 1 - \frac{x^2}{r_1^2}. \quad (18)$$

Furthermore,

$$q = 0. \quad (19)$$

From equations (10) – (19) it follows that

$$\sigma_{pl} = 0. \quad (20)$$

We are looking for the strains, ε_{ml} , in the outer arm of the crack. The thin-walled elliptical cylinder has non-linear elastic behavior. The relationship between stresses and strains obeys the following non-linear constitutive law (Lukash, 1997):

$$\sigma_{ml} = E\varepsilon_{ml} - B\varepsilon_{ml}^s - D\varepsilon_{ml}^f, \quad (21)$$

where E , B , D , s and f are material parameters. Since the wall of the cylinder has a continuous material inhomogeneity along the height, the parameters, E , B , D , s and f , vary along the axis, z , according to the laws

$$E = E_\alpha + \frac{E_\beta - E_\alpha}{h^\lambda} z^\lambda, \quad (22)$$

$$B = B_\alpha + \frac{B_\beta - B_\alpha}{h^\mu} z^\mu, \quad (23)$$

$$D = D_\alpha + \frac{D_\beta - D_\alpha}{h^\eta} z^\eta, \quad (24)$$

$$s = s_\alpha + \frac{s_\beta - s_\alpha}{h^\nu} z^\nu, \quad (25)$$

$$f = f_\alpha + \frac{f_\beta - f_\alpha}{h^\psi} z^\psi, \quad (26)$$

$$0 \leq z \leq h, \quad (27)$$

where E_α , B_α , D_α , s_α and f_α are the values of E , B , D , s and f at the lower end of the cylinder, E_β , B_β , D_β , s_β and f_β are the corresponding values at the upper end of the cylinder, λ , μ , η , ν and ψ are parameters.

The strain, ε_{ml} , is found from (21) for different values of z in the interval (27). Due to the specifics of the cylinder loading, the strains, ε_{md} , in the inner crack arm are equal to ε_{ml} . To find the strains, ε_m , in the intact section of the cylinder, we use relation (21). To do this, we replace σ_{ml} and ε_{ml} with σ_m and ε_m and then find ε_m for z varying in the interval (9).

We determine the SERR, G , by formula

$$G = \frac{dU^*}{dA}, \quad (28)$$

where

$$dA = cda. \quad (29)$$

The circumference of the ellipse is

$$c = \pi \left[3(r_1 + r_2) - \sqrt{(3r_1 + r_2)(r_1 + 3r_2)} \right]. \quad (30)$$

The complementary strain energy, U^* , involved in equation (29) is found as

$$U^* = 2U_{ml}^* + U_m^*, \quad (31)$$

where U_{ml}^* and U_m^* are the complementary strain energies in the outer crack arm and in the intact portion of the cylinder, respectively. The complementary strain energy in the outer arm is multiplied by two in (31) because the complementary strain energies in the two crack arms are equal. The complementary strain energy in the outer arm is

$$U_{ml}^* = \frac{b}{2} c \int_0^a u_{0ml}^* dz, \quad (32)$$

where u_{0ml}^* is the specific complementary strain energy. It is determined by the formula

$$u_{0ml}^* = \sigma_{ml} \varepsilon_{ml} - \int_0^{\varepsilon_{ml}} \sigma_{ml} d\varepsilon_{ml}. \quad (33)$$

We substitute (21) into (33) and obtain

$$u_{0ml}^* = \frac{E}{2} \varepsilon_{ml}^2 - \frac{Bs}{s+1} \varepsilon_{ml}^{s+1} - \frac{Df}{f+1} \varepsilon_{ml}^{f+1}. \quad (34)$$

The complementary strain energy in the intact portion of the cylinder is

$$U_m^* = bc \int_a^h u_{0m}^* dz. \quad (35)$$

The specific complementary strain energy, u_{0m}^* is determined by replacing ε_{ml} with ε_m in (34). Then, the SERR is found by formula (28). To verify the thus obtained solution for the SERR, an existing approach is applied, according to which the SERR is found as (Rizov, 2017)

$$G = bcu_{0mlbh}^* - bcu_{0mfr}^*, \quad (36)$$

where u_{0mlbh}^* and u_{0mfr}^* are the specific complementary strain energies in the cylinder cross-sections behind and in ahead of the crack front. The use of (36) leads to results that match those obtained with (28).

The SERR is also derived for the case when the thin-walled elliptical cylinder performs a uniform rotation about the axis, z , with angular velocity, ω , as shown in Fig. 3. In this case, the cylinder has both meridional stress from its own weight and hoop stress from the inertial load generated by the normal acceleration due to uniform rotation around the axis, z .

First, the meridional stress, $\sigma_{ml\omega}$, in the outer arm of the crack is found. For this purpose,

$$a_v = 0 \quad (37)$$

is substituted into equation (5). The result is

$$\sigma_{ml\omega} = \gamma z g. \quad (38)$$

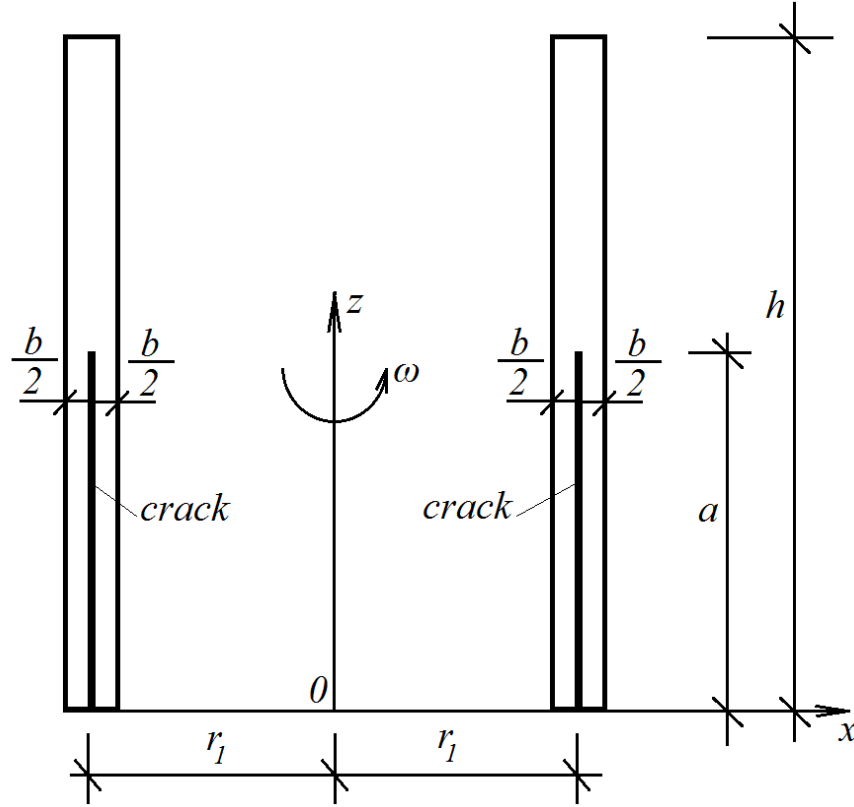


Figure 3. Thin-walled elliptical cylinder rotating around the axis, z

Then, the meridional stress, σ_{ml} , is found in the intact portion of the cylinder by substituting (37) in (8). This leads again to expression (38). However, in this case

$$a \leq z \leq h. \quad (39)$$

We seek the hoop stress, $\sigma_{pl\omega}$, in the outer arm of the crack. In this regard, we first find the inertial load component, q , generated by the normal acceleration, a_n .

$$q = \gamma \frac{b}{2} a_n \cos \theta_t, \quad (40)$$

where

$$a_n = \omega^2 OT, \quad (41)$$

$$\theta_t = -\frac{\pi}{2} + \arctg \theta_\tau - \arctg \left(\frac{r_2}{x} \sqrt{\theta_\kappa} \right). \quad (42)$$

The distance, OT , is shown in Fig. 2. It is found as

$$OT = \sqrt{x^2 + r_2^2 \theta_\kappa}. \quad (43)$$

We write the Laplace equation for the outer crack arm

$$\frac{\sigma_{ml\omega}}{\rho_m} + \frac{\sigma_{pl\omega}}{\rho_p} = \frac{q}{\frac{b}{2}}. \quad (44)$$

From (44) we obtain

$$\sigma_{pl\omega} = \rho_p \frac{q}{\frac{b}{2}}. \quad (45)$$

The hoop stress, $\sigma_{pd\omega}$, in the inner arm of the crack is equal to $\sigma_{pl\omega}$, since the cylinder under consideration is thin-walled. The hoop stress, $\sigma_{p\omega}$, in the intact portion of the cylinder is found by replacing $b/2$ with b in formula (45).

We are looking for the meridional strain, $\varepsilon_{ml\omega}$, in the outer arm of the crack. We apply the relation (21). For this purpose, we replace σ_{ml} with $\sigma_{ml\omega}$ and then solve the equation thus obtained with respect to $\varepsilon_{ml\omega}$ using the MatLab. The strain along the meridian in the inner arm of the crack, $\varepsilon_{md\omega}$, is equal to $\varepsilon_{ml\omega}$.

To determine $\varepsilon_{m\omega}$ in the intact portion of the cylinder, we use (21) again. For this purpose, we replace σ_{ml} and ε_{ml} with $\sigma_{m\omega}$ and $\varepsilon_{m\omega}$ and then solve the equation with respect to $\varepsilon_{m\omega}$.

The hoop strain, $\varepsilon_{pl\omega}$, in the outer arm of the crack is determined from (21). To do this, we replace σ_{ml} and ε_{ml} with $\sigma_{pl\omega}$ and $\varepsilon_{pl\omega}$ and then solve the equation for $\varepsilon_{pl\omega}$. The hoop strain, $\varepsilon_{pd\omega}$, in the inner arm of the crack is equal to $\varepsilon_{pl\omega}$.

The hoop strain, $\varepsilon_{p\omega}$, in the intact portion of the cylinder is found by replacing σ_{ml} and ε_{ml} with $\sigma_{p\omega}$ and $\varepsilon_{p\omega}$ in (21) and then solve the equation in terms of $\varepsilon_{p\omega}$.

Finally, to find the SERR formula (28) is applied. For this purpose, both the meridional and hoop stresses in both arms of the crack and in the intact portion of the elliptical cylinder are taken into account when determining the complementary strain energy. To check the SERR, (36) is again applied.

Results and Discussion

The most important result of this paper is that the SERR solutions have been obtained for a lengthwise crack in a moving thin-walled continuously inhomogeneous elliptical cylinder with non-linear elastic behavior. The crack represents an elliptical surface coinciding with the middle surface of the cylinder. The cylinder itself is under the action of its own weight and the inertial load from the acceleration. In principle, the paper considers two types of mechanical motion of the thin-walled elliptical cylinder, namely:

- 1) translational motion in the vertical direction with constant acceleration;
- 2) uniform rotation around the axis, z .

The SERR solutions have been obtained for both types of motion. The solutions are presented in detail with all necessary explanations and diagrams of the cylinder. All necessary formulas are derived, which allows for direct application of the solutions in the analysis of moving elliptical cylinders with a lengthwise crack in them. As a result of continuous material inhomogeneity, the parameters of the non-linear constitutive law, as well as the bulk density of the material, change continuously along the height of the cylinder. The approach described in the

paper can also be applied when using other constitutive laws for non-linear elastic behavior of the cylinder and other laws for changing material parameters along the height.

Conclusion

It can be summarized that the analysis presented in this paper has the potential for practical application in the study of moving thin-walled elliptical cylinders with a lengthwise crack (two motions are considered, namely translational motion in the vertical direction and uniform rotation). The analysis is relatively general, because it takes into account the non-linear elastic behavior of the material from which the cylinder wall is made. In addition, the material is continuously inhomogeneous along the height of the cylinder. It is necessary to emphasize here that the obtained solutions of the SERR apply not only to cylinders in which the parameters of the non-linear constitutive law change continuously along the height, but also to cylinders in which the bulk density of the material is also a continuous function of the coordinate, z . The approach presented in the paper is also applicable to obtaining SERR in cylinders with linear-elastic behavior. For this purpose, Hooke's law should be applied instead of non-linear constitutive law. For example, this can be realized by setting $B=0$ and $D=0$ in law (21). Then, from the non-linear law (21), Hooke's law is obtained. This means that if we put $B=0$ and $D=0$ in the solution, we will obtain the SERR in a cylinder with linear-elastic behavior. The approach can also be applied to find the SERR in homogeneous cylinders. For this purpose, the resulting solution must assume equal values of the material parameters at both end of the cylinder (then the parameters, γ , E , B , D , s and f , will not change along the height (refer to formulas (3), (22) – (26))).

Recommendations

The analysis developed in this paper can be recommended as a means of studying the lengthwise fracture of moving thin-walled elliptical cylinders exhibiting non-linear elastic behavior.

Scientific Ethics Declaration

* The author declares that the scientific ethical and legal responsibility of this article published in EPSTEM Journal belongs to the author.

Conflict of Interest

* The author declares that there is no conflict of interest.

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